Relative topos theory via stacks: an introduction

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Plan of the course

- Motivation
- Topos-theoretic background
- Toposes as 'bridges'
- Functors inducing morphisms of toposes
- Relative toposes (joint work with Riccardo Zanfa)
  - Relative presheaf toposes
  - The fundamental adjunction
  - Relative sheaf toposes
- A problem of Grothendieck

Relativity techniques

- Broadly speaking, in Mathematics the relativization method consists in trying to state notions and results in terms of morphisms, rather than objects, of a given category, so that they can be 'relativized' to an arbitrary base object.
- One works in the new, relative universe as it were the 'classical' one, and then interprets the obtained results from the point of view of the original universe. This process is usually called *externalization*.
- Relativity techniques can be thought as general 'change of base techniques', allowing one to choose the universe relatively to which one works according to one's needs.
- The relativity method has been pioneered by Grothendieck, in particular for schemes, in his categorical refoundation of Algebraic Geometry, and have played a key role in his work.
- We aim for a similar set of tools for toposes, that is, for an efficient formalism for doing topos theory over an arbitrary base topos.

Topos theory over an arbitrary base topos

Our new foundations for relative topos theory are based on stacks (and, more generally, fibrations and indexed categories).

The approach of category theorists (Lawvere, Diaconescu, Johnstone, etc.) to this subject is chiefly based on the notions of internal category and of internal site.

The problem with these notions is that they are too rigid to naturally capture relative topos-theoretic phenomena, as well as for making computations and formalizing 'parametric reasoning'.

We shall resort to the more general and technically flexible notion of stack, developing the point of view originally introduced by J. Giraud in his paper *Classifying topos*.

Grothendieck toposes

- The notion of topos was introduced in the early sixties by A. Grothendieck with the aim of bringing a topological or geometric intuition also in areas where actual topological spaces do not occur.
- Grothendieck realized that many important properties of topological spaces X can be naturally formulated as (invariant) properties of the categories $\mathbf{Sh}(X)$ of sheaves of sets on the spaces.
- He then defined toposes as more general categories of sheaves of sets, by replacing the topological space X by a (small) site, that is a pair $(\mathcal{C}, J)$ consisting of a (small) category $\mathcal{C}$ and a 'generalized notion of covering' J on it, and taking sheaves (in a generalized sense) over the site:

$$\begin{array}{ccc} X & \dashrightarrow & \mathbf{Sh}(X) \\ \rightsquigarrow & & \rightsquigarrow \\ (\mathcal{C}, J) & \dashrightarrow & \mathbf{Sh}(\mathcal{C}, J) \end{array}$$

Sieves

The notion of Grothendieck topology on a category represents a 'categorification' of the classical notion of covering of an open set of a topological space by a family of open subsets. In order to define it in full generality, one needs to talk about *sieves*.

Definition

Given a category $\mathcal{C}$ and an object $c \in \mathrm{Ob}(\mathcal{C})$, a sieve S in $\mathcal{C}$ on c is a collection of arrows in $\mathcal{C}$ with codomain c such that

$$f \in S \Rightarrow f \circ g \in S$$

whenever this composition makes sense.

If S is a sieve on c and $h : d \to c$ is any arrow to c, then

$$h^*(S) := \{g \mid \mathrm{cod}(g) = d,\ h \circ g \in S\}$$

is a sieve on d.

Remark

Sieves in a category $\mathcal{C}$ on an object c correspond precisely to the subobjects of the representable functor $\mathrm{Hom}_{\mathcal{C}}(-, c)$ in the category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ of presheaves on $\mathcal{C}$.

Grothendieck topologies

Definition

A Grothendieck topology on a category $\mathcal{C}$ is a function J which assigns to each object c of $\mathcal{C}$ a collection $J(c)$ of sieves on c in such a way that

(i) (maximality axiom) the maximal sieve $M_c = \{f \mid cod(f) = c\}$ is in $J(c)$;

(ii) (stability axiom) if $S \in J(c)$, then $f^*(S) \in J(d)$ for any arrow $f : d \to c$;

(iii) (transitivity axiom) if $S \in J(c)$ and R is any sieve on c such that $f^*(R) \in J(d)$ for all $f : d \to c$ in S, then $R \in J(c)$.

The sieves S which belong to $J(c)$ for some object c of $\mathcal{C}$ are said to be J-covering.

Sites

- A site (resp. small site) is a pair $(\mathcal{C}, J)$ where $\mathcal{C}$ is a category (resp. a small category) and J is a Grothendieck topology on $\mathcal{C}$.

- A site $(\mathcal{C}, J)$ is said to be small-generated if $\mathcal{C}$ is locally small and has a small J-dense subcategory (that is, a category $\mathcal{D}$ such that every object of $\mathcal{C}$ admits a J-covering sieve generated by arrows whose domains lie in $\mathcal{D}$, and for every arrow $f : d \to c$ in $\mathcal{C}$ where d lies in $\mathcal{D}$ the family of arrows $g : \mathrm{dom}(g) \to d$ such that $f \circ g$ lies in $\mathcal{D}$ generates a J-covering sieve).

Remark

It is important to allow oneself to work with small-generated sites, rather than merely with small sites, for greater generality and technical flexibility.

Matching families and amalgamations

Definition

- A presheaf on a (small) category $\mathcal{C}$ is a functor $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$.
- Let $P : \mathcal{C}^{\text{op}} \to \mathbf{Set}$ be a presheaf on $\mathcal{C}$ and S be a sieve on an object c of $\mathcal{C}$.

  A matching family for S of elements of P is a function which assigns to each arrow $f : d \to c$ in S an element $x_f \in P(d)$ in such a way that

  $$P(g)(x_f) = x_{f \circ g} \quad \text{for all } g : e \to d\ .$$

  An amalgamation for such a family is a single element $x \in P(c)$ such that

  $$P(f)(x) = x_f \quad \text{for all } f \text{ in } S\ .$$

Remark

Matching families for S of elements of P correspond precisely to natural transformations $S \to P$, that is, to morphisms $S \to P$ in the presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$, where S is regarded as a subobject of the representable $\mathrm{Hom}_{\mathcal{C}}(-, c)$, while amalgamations correspond to morphisms $\mathrm{Hom}_{\mathcal{C}}(-, c) \to P$ (by the Yoneda lemma).

Sheaves on a site

- Given a site $(\mathcal{C}, J)$, a presheaf on $\mathcal{C}$ is a J-sheaf if every matching family for any J-covering sieve on any object of $\mathcal{C}$ has a unique amalgamation.
- The category $\mathbf{Sh}(\mathcal{C}, J)$ of sheaves on the site $(\mathcal{C}, J)$ is the full subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ on the presheaves which are J-sheaves.

Definition

A Grothendieck topos is any category equivalent to the category of sheaves on a small (or equivalently, small-generated) site.

Geometric morphisms

The natural, topologically motivated, notion of morphism of Grothendieck toposes is that of geometric morphism. The natural notion of morphism of geometric morphisms if that of geometric transformation.

Definition

(i) Let $\mathcal{E}$ and $\mathcal{F}$ be toposes. A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ consists of a pair of functors $f_* : \mathcal{E} \to \mathcal{F}$ (the direct image of f) and $f^* : \mathcal{F} \to \mathcal{E}$ (the inverse image of f) together with an adjunction $f^* \dashv f_*$, such that f^* preserves finite limits.

(ii) Let f and $g : \mathcal{E} \to \mathcal{F}$ be geometric morphisms. A geometric transformation $\alpha : f \to g$ is defined to be a natural transformation $a : f^* \to g^*$.

(iii) A point of a topos $\mathcal{E}$ is a geometric morphism $\mathbf{Set} \to \mathcal{E}$.

- Grothendieck toposes and geometric morphisms between them form a 2-category.
- Given two toposes $\mathcal{E}$ and $\mathcal{F}$, geometric morphisms from $\mathcal{E}$ to $\mathcal{F}$ and geometric transformations between them form a category, denoted by $\mathbf{Geom}(\mathcal{E}, \mathcal{F})$.

Examples of geometric morphisms

- A continuous function $f : X \to Y$ between topological spaces gives rise to a geometric morphism $\mathbf{Sh}(f) : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$. The direct image $\mathbf{Sh}(f)_*$ sends a sheaf $F \in Ob(\mathbf{Sh}(X))$ to the sheaf $\mathbf{Sh}(f)_*(F)$ defined by $\mathbf{Sh}(f)_*(F)(V) = F(f^{-1}(V))$ for any open subset V of Y. The inverse image $\mathbf{Sh}(f)^*$ acts on étale bundles over Y by sending an étale bundle $p : E \to Y$ to the étale bundle over X obtained by pulling back p along $f : X \to Y$.
- Every Grothendieck topos $\mathcal{E}$ has a unique geometric morphism $\mathcal{E} \to \mathbf{Set}$. The direct image is the global sections functor $\Gamma : \mathcal{E} \to \mathbf{Set}$, sending an object $e \in \mathcal{E}$ to the set $\mathrm{Hom}_{\mathcal{E}}(1_{\mathcal{E}}, e)$, while the inverse image functor $\Delta : \mathbf{Set} \to \mathcal{E}$ sends a set S to the coproduct $\coprod_{s \in S} 1_{\mathcal{E}}$.
- For any site $(\mathcal{C}, J)$, the pair of functors formed by the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and the associated sheaf functor $a : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$ yields a geometric morphism $i : \mathbf{Sh}(\mathcal{C}, J) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.
- For any Grothendieck topos $\mathcal{E}$ and any morphism $f : P \to Q$ in $\mathcal{E}$, the pullback functor $f^* : \mathcal{E}/Q \to \mathcal{E}/P$ has both a left adjoint (namely, the functor Σ_f given by composition with f) and a right adjoint π_f. It is therefore the inverse image of a geometric morphism $\mathcal{E}/P \to \mathcal{E}/Q$.

A general hom-tensor adjunction I

Theorem

Let $\mathcal{C}$ be a small category, $\mathcal{E}$ be a locally small cocomplete category and $A : \mathcal{C} \to \mathcal{E}$ a functor. Then we have an adjunction

$$L_A : [\mathcal{C}^{\text{op}}, \textbf{Set}] \rightleftarrows \mathcal{E} : R_A$$

where the right adjoint $R_A : \mathcal{E} \to [\mathcal{C}^{\text{op}}, \textbf{Set}]$ is defined for each $e \in Ob(\mathcal{E})$ and $c \in Ob(\mathcal{C})$ by:

$$R_A(e)(c) = \text{Hom}_{\mathcal{E}}(A(c), e)$$

and the left adjoint $L_A : [\mathcal{C}^{\text{op}}, \textbf{Set}] \to \mathcal{E}$ is defined by

$$L_A(P) = \text{colim}(A \circ \pi_P),$$

where π_P is the canonical projection functor $\int P \to \mathcal{C}$ from the category of elements $\int P$ of P to $\mathcal{C}$.

A general hom-tensor adjunction II

Remarks

- *The functor L_A can be considered as a generalized tensor product, since, by the construction of colimits in terms of coproducts and coequalizers, we have the following coequalizer diagram:*

$$\coprod_{\substack{c\in\mathcal{C}, p\in P(c) \\ u:c'\to c}} A(c') \underset{\tau}{\overset{\theta}{\rightrightarrows}} \coprod_{c\in\mathcal{C},\, p\in P(c)} A(c) \xrightarrow{\phi} L_A(P),$$

  where

$$\theta(c,p,u,x) = (c', P(u)(p), x)$$

  and

$$\tau(c,p,u,x) = (c,p,A(u)(x))\ .$$

  For this reason, we shall also denote L_A by

$$-\otimes_{\mathcal{C}} A : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathcal{E}\ .$$

- *We can rewrite the above coequalizer as follows:*

$$\coprod_{c,c'\in\mathcal{C}} P(c)\times \mathrm{Hom}_{\mathcal{C}}(c',c)\times A(c') \underset{\tau}{\overset{\theta}{\rightrightarrows}} \coprod_{c\in\mathcal{C}} P(c)\times A(c) \xrightarrow{\phi} P\otimes_{\mathcal{C}} A\ .$$

  From this we see that this definition is symmetric in P and A, that is

$$P\otimes_{\mathcal{C}} A \cong A\otimes_{\mathcal{C}^{\text{op}}} P\ .$$

Geometric morphisms as flat functors I

Definition

- A functor $A : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to a locally small topos $\mathcal{E}$ with small colimits is said to be flat if the functor $- \otimes_{\mathcal{C}} A : [\mathcal{C}^{\text{op}}, \textbf{Set}] \to \mathcal{E}$ preserves finite limits.
- The full subcategory of $[\mathcal{C}, \mathcal{E}]$ on the flat functors will be denoted by $\textbf{Flat}(\mathcal{C}, \mathcal{E})$.

Theorem

Let $\mathcal{C}$ be a small category and $\mathcal{E}$ be a Grothendieck topos. Then we have an equivalence of categories

$$\textbf{Geom}(\mathcal{E}, [\mathcal{C}^{\text{op}}, \textbf{Set}]) \simeq \textbf{Flat}(\mathcal{C}, \mathcal{E})$$

(natural in $\mathcal{E}$), which sends

- *a flat functor $A : \mathcal{C} \to \mathcal{E}$ to the geometric morphism $\mathcal{E} \to [\mathcal{C}^{\text{op}}, \textbf{Set}]$ determined by the functors R_A and $- \otimes_{\mathcal{C}} A$, and*
- *a geometric morphism $f : \mathcal{E} \to [\mathcal{C}^{\text{op}}, \textbf{Set}]$ to the flat functor given by the composite $f^* \circ y_{\mathcal{C}}$ of $f^* : [\mathcal{C}^{\text{op}}, \textbf{Set}] \to \mathcal{E}$ with the Yoneda embedding $y_{\mathcal{C}} : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \textbf{Set}]$.*

Flat = filtering

Definition

A functor $F : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to a Grothendieck topos $\mathcal{E}$ is said to be filtering if it satisfies the following conditions:

(a) For any object E of $\mathcal{E}$ there exist an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each $i \in I$ an object b_i of $\mathcal{C}$ and a generalized element $E_i \to F(b_i)$ in $\mathcal{E}$.

(b) For any two objects c and d in $\mathcal{C}$ and any generalized element $\langle x, y\rangle : E \to F(c) \times F(d)$ in $\mathcal{E}$ there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each $i \in I$ an object b_i of $\mathcal{C}$ with arrows $u_i : b_i \to c$ and $v_i : b_i \to d$ in $\mathcal{C}$ and a generalized element $z_i : E_i \to F(b_i)$ in $\mathcal{E}$ such that $\langle F(u_i), F(v_i)\rangle \circ z_i = \langle x, y\rangle \circ e_i$ for all $i \in I$.

(c) For any two parallel arrows $u, v : d \to c$ in $\mathcal{C}$ and any generalized element $x : E \to F(d)$ in $\mathcal{E}$ for which $F(u) \circ x = F(v) \circ x$, there is an epimorphic family $\{e_i : E_i \to E \mid i \in I\}$ in $\mathcal{E}$ and for each $i \in I$ an arrow $w_i : b_i \to d$ and a generalized element $y_i : E_i \to F(b_i)$ such that $u \circ w_i = v \circ w_i$ and $F(w_i) \circ y_i = x \circ e_i$ for all $i \in I$.

Theorem (Mac Lane and Moerdijk)

A functor $F : \mathcal{C} \to \mathcal{E}$ from a small category $\mathcal{C}$ to a Grothendieck topos $\mathcal{E}$ is flat if and only if it is filtering.

Remarks

- *For any small category $\mathcal{C}$, a functor $P : \mathcal{C} \to$ **Set** is filtering if and only if its category of elements $\int P$ is a filtered category (equivalently, if it is a filtered colimit of representables).*
- *For any small cartesian category $\mathcal{C}$, a functor $\mathcal{C} \to \mathcal{E}$ is flat if and only if it preserves finite limits.*

Geometric morphisms to $\mathbf{Sh}(\mathcal{C}, J)$

Definition

If $(\mathcal{C}, J)$ is a site, a flat functor $F : \mathcal{C} \to \mathcal{E}$ to a Grothendieck topos is said to be *J*-continuous if it sends J-covering sieves to epimorphic families.

The full subcategory of $\mathbf{Flat}(\mathcal{C}, \mathcal{E})$ on the J-continuous flat functors will be denoted by $\mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$.

Theorem

For any site $(\mathcal{C}, J)$ and Grothendieck topos $\mathcal{E}$, the above-mentioned equivalence between geometric morphisms and flat functors restricts to an equivalence of categories

$$\mathbf{Geom}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq \mathbf{Flat}_J(\mathcal{C}, \mathcal{E})$$

natural in $\mathcal{E}$.

Sketch of proof.

Appeal to the previous theorem

- identifying the geometric morphisms $\mathcal{E} \to \mathbf{Sh}(\mathcal{C}, J)$ with the geometric morphisms $\mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ which factor through the canonical geometric inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$, and
- using the characterization of such morphisms as the geometric morphisms $f : \mathcal{E} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ such that the composite $f^* \circ y$ of the inverse image functor f^* of f with the Yoneda embedding $y : \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ sends J-covering sieves to epimorphic families in $\mathcal{E}$.

Morphisms and comorphisms of sites

Geometric morphisms can be naturally induced by functors between sites satisfying appropriate properties:

Definition

- A morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that the composite $l' \circ F$, where l' is the canonical functor $\mathcal{D} \to \mathbf{Sh}(\mathcal{D}, K)$, is flat and J-continuous. If $\mathcal{C}$ and $\mathcal{D}$ have finite limits then F is a morphism of sites if it preserves finite limits and is cover-preserving.
- A comorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$ is a functor $\pi : \mathcal{D} \to \mathcal{C}$ which has the covering-lifting property (in the sense that for any $d \in \mathcal{D}$ and any J-covering sieve S on $\pi(d)$ there is a K-covering sieve R on d such that $\pi(R) \subseteq S$).

We have the following well-known fundamental result, which we shall discuss in detail below:

Theorem

- *Every morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ induces a geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.*
- *Every comorphism of sites $\pi : (\mathcal{D}, K) \to (\mathcal{C}, J)$ induces a geometric morphism $C_\pi : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.*

Characterizing morphisms of sites

We can explicitly characterize the functors which are morphisms of sites by using:

- the characterization of filtering functors with values in a Grothendieck topos as functors which send certain families to epimorphic families,
- the fact that the image under the associated sheaf functor of a family of natural transformations with common codomain is epimorphic if and only if the family is locally jointly surjective, and
- the following description of the arrows in a Grothendieck topos between objects coming from a site in terms of locally compatible families of arrows in the site.

Arrows in a Grothendieck topos

Given a site $(\mathcal{C}, J)$, for two arrows $h, k : c \to d$ in $\mathcal{C}$ we shall write $h \equiv_J k$ for J-local equality, that is, to mean that there exists a J-covering sieve S on c such that $h \circ f = k \circ f$ for every $f \in S$. Notice that, denoting by l the canonical functor $\mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$, $l(h) = l(k)$ if and only if $h \equiv_J k$.

Proposition

Let $(\mathcal{C}, J)$ be a small-generated site.

(i) *Then any arrow $\xi : l(c) \to l(d)$ in $\mathbf{Sh}(\mathcal{C}, J)$ admits a local representation by a family of arrows $\{f_u : c_u \to c, g_u : c_u \to d \mid u \in U\}$ such that $\{f_u : c_u \to c \mid u \in U\}$ generates a J-covering sieve, for any object e and arrows $h : e \to c_u$ and $k : e \to c_{u'}$ such that $f_u \circ h = f_{u'} \circ k$ we have $g_u \circ h \equiv_J g_{u'} \circ k$, and $\xi \circ l(f_u) = l(g_u)$ for every $u \in U$.*

(ii) *Conversely, any family $\mathcal{F} : \{f_u : c_u \to c, g_u : c_u \to d \mid u \in U\}$ such that $\{f_u : c_u \to c \mid u \in U\}$ generates a J-covering sieve and for any object e and arrows $h : e \to c_u$ and $k : e \to c_{u'}$ such that $f_u \circ h = f_{u'} \circ k$ we have $g_u \circ h \equiv_J g_{u'} \circ k$, determines a unique arrow $\xi_{\mathcal{F}} : l(c) \to l(d)$ in $\mathbf{Sh}(\mathcal{C}, J)$ such that $\xi_{\mathcal{F}} \circ l(f_u) = l(g_u)$ for every $u \in U$.*

Arrows in a Grothendieck topos

(iii) Two families $\mathcal{F} = \{f_u : c_u \to c, g_u : c_u \to d \mid u \in U\}$ and $\mathcal{F}' = \{f'_v : e_v \to c, g'_v : e_v \to d \mid v \in V\}$ as in (ii) determine the same arrow $l(c) \to l(d)$ (i.e. $\xi_{\mathcal{F}} = \xi_{\mathcal{F}'}$) if and only if they are locally equal on a common refinement, i.e. if there exist a J-covering family $\{a_k : b_k \to c \mid k \in K\}$ and factorizations of it through both of them by arrows $x_k : b_k \to c_{u(k)}$ and $y_k : b_k \to e_{v(k)}$ (i.e. $f_{u(k)} \circ x_k = a_k = f'_{v(k)} \circ y_k$ for every $k \in K$) such that $g_{u(k)} \circ x_k \equiv_J g'_{v(k)} \circ y_k$ for every $k \in K$.

(iv) Given two families $\mathcal{F} = \{f_u : c_u \to c, g_u : c_u \to d \mid u \in U\}$ and $\mathcal{G} = \{h_v : d_v \to d, k_v : d_v \to e \mid v \in V\}$, the composite arrow $\xi_{\mathcal{G}} \circ \xi_{\mathcal{F}} : l(c) \to l(e)$ is induced as in (ii) by the family $\{f_u \circ x : \mathrm{dom}(x) \to c, k_v \circ y : \mathrm{dom}(y) \to e \mid (u, v, x, y) \in Z\}$, where $Z = \{(u, v, x, y) \mid u \in U, v \in V, \mathrm{dom}(x) = \mathrm{dom}(y), \mathrm{cod}(x) = c_u, \mathrm{cod}(y) = d_v, h_v \circ y = g_u \circ x\}$.

Arrows in a Grothendieck topos

Proposition

Let $(\mathcal{C}, J)$ be a small-generated site and a_J the associated sheaf functor $[\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$. Then

1. *An arrow $\xi : l(c) \to a_J(P)$ in $\mathbf{Sh}(\mathcal{C}, J)$ (equivalently, an element of $a_J(P)(c)$) can be identified with an equivalence class of families $\{x_f \in P(\mathrm{dom}(f)) \mid f \in S\}$ of elements of P indexed by the arrows f of a J-covering sieve S on c which are locally matching in the sense that for any arrow g composable with an arrow $f \in S$, $x_{f \circ g} \equiv_J P(g)(x_f)$, modulo the equivalence which identifies two such families when they are locally equal on a common refinement.*
2. *Any such family yields a local representation of ξ in the sense that $\xi \circ l(f) = r_{x_f}$ for each $f \in S$, where r_{x_f} is the image under a_J of the arrow $y_{\mathcal{C}}(\mathrm{dom}(f)) \to P$ corresponding to the element $x_f \in P(\mathrm{dom}(f))$ via the Yoneda lemma.*

Remark

The proposition gives an explicit description of the associated sheaf functor $a_J(P)$ of a presheaf P, different from the usual construction of it by means of the double plus construction. This alternative construction of the associated sheaf functor seems to have been first discovered (albeit not published) by Eduardo Dubuc in the eighties.

J-functional relations

More generally, for any presheaves $P, Q \in [\mathcal{C}^{\text{op}}, \mathbf{Set}]$, the arrows $a_J(P) \to a_J(Q)$ in $\mathbf{Sh}(\mathcal{C}, J)$ are in natural bijection with the J-functional relations from P to Q in $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$, in the sense of the following

Definition

In a presheaf topos $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$, a relation $R \rightarrowtail P \times Q$ (that is, an assignment $c \to R(c)$ to each object c of $\mathcal{C}$ of a subset $R(c)$ of $P(c) \times Q(c)$ which is *functorial* in the sense that for any arrow $f : c \to c'$ in $\mathcal{C}$, $P(f) \times Q(f)$ sends $R(c')$ to $R(c)$), is said to be J-functional from P to Q if it satisfies the following properties:

(i) for any $c \in \mathcal{C}$ and any $(x, y) \in P(c) \times Q(c)$, if $\{f : d \to c \mid (P(f)(x), Q(f)(y)) \in R(d)\} \in J(c)$ then $(x, y) \in R(c)$;

(ii) for any $c \in \mathcal{C}$ and any $(x, y), (x', y') \in R(c)$, if $x = x'$ then $\{f : d \to c \mid Q(f)(y) = Q(f)(y')\} \in J(c)$;

(iii) for any $c \in \mathcal{C}$ and any $x \in P(c)$, $\{f : d \to c \mid \exists y \in Q(d)\, (P(f)(x), y) \in R(d)\} \in J(c)$.

Morphisms of sites

Theorem

Let $(\mathcal{C}, J)$ and $(\mathcal{C}', J')$ be small-generated sites, and $l : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$, $l' : \mathcal{C}' \to \mathbf{Sh}(\mathcal{C}', J')$ be the canonical functors (given by the composite of the relevant Yoneda embedding with the associated sheaf functor). Then, given a functor $F : \mathcal{C} \to \mathcal{C}'$, the following conditions are equivalent:

(i) *A induces a geometric morphism $u : \mathbf{Sh}(\mathcal{C}', J') \to \mathbf{Sh}(\mathcal{C}, J)$ making the following square commutative:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle l'} \\ \mathbf{Sh}(\mathcal{C}, J) & \xrightarrow{u^*} & \mathbf{Sh}(\mathcal{C}', J'); \end{array}$$

(ii) *The functor F is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{C}', J')$ in the sense that it satisfies the following properties:*

(1) *A sends every J-covering family in $\mathcal{C}$ into a J'-covering family in $\mathcal{C}'$.*

(2) *Every object c' of $\mathcal{C}'$ admits a J'-covering family*

$$c'_i \longrightarrow c', \quad i \in I,$$

by objects c'_i of $\mathcal{C}'$ which have morphisms

$$c'_i \longrightarrow F(c_i)$$

to the images under A of objects c_i of $\mathcal{C}$.

Morphisms of sites

(3) For any objects c_1, c_2 of $\mathcal{C}$ and any pair of morphisms of $\mathcal{C}'$

$$f_1' : c' \longrightarrow F(c_1)\,, \quad f_2' : c' \longrightarrow F(c_2)\,,$$

there exists a J'-covering family

$$g_i' : c_i' \longrightarrow c'\,, \quad i \in I\,,$$

and a family of pairs of morphisms of $\mathcal{C}$

$$f_1^i : b_i \longrightarrow c_1\,, \quad f_2^i : b_i \to c_2\,, \quad i \in I\,,$$

and of morphisms of $\mathcal{C}'$

$$h_i' : c_i' \longrightarrow F(b_i)\,, \quad i \in I\,,$$

making the following squares commutative:

$$\begin{array}{ccc} c_i' & \xrightarrow{g_i'} & c' \\ {\scriptstyle h_i'}\downarrow & & \downarrow{\scriptstyle f_1'} \\ F(b_i) & \xrightarrow{F(f_1^i)} & F(c_1) \end{array} \qquad \begin{array}{ccc} c_i' & \xrightarrow{g_i'} & c' \\ {\scriptstyle h_i'}\downarrow & & \downarrow{\scriptstyle f_2'} \\ F(b_i) & \xrightarrow{F(f_2^i)} & F(c_2) \end{array}$$

Morphisms of sites

(4) For any pair of arrows $f_1, f_2 : c \rightrightarrows d$ of $\mathcal{C}$ and any arrow of $\mathcal{C}'$

$$f' : b' \longrightarrow F(c)$$

satisfying

$$F(f_1) \circ f' = F(f_2) \circ f' \,,$$

there exist a J'-covering family

$$g'_i : b'_i \longrightarrow b' \,, \quad i \in I \,,$$

and a family of morphisms of $\mathcal{C}$

$$h_i : b_i \longrightarrow c \,, \quad i \in I \,,$$

satisfying

$$f_1 \circ h_i = f_2 \circ h_i \,, \quad \forall i \in I \,,$$

and of morphisms of $\mathcal{C}'$

$$h'_i : b'_i \longrightarrow F(b_i) \,, \quad i \in I \,,$$

making commutative the following squares:

$$\begin{array}{ccc} b'_i & \xrightarrow{g'_i} & b' \\ {\scriptstyle h'_i}\downarrow & & \downarrow{\scriptstyle f'} \\ F(b_i) & \xrightarrow{F(h_i)} & F(c) \end{array}$$

Morphisms of sites

If F is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$, we denote by $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ the geometric morphism which it induces.

Remarks

- *If $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ are cartesian sites (that is, $\mathcal{C}$ and $\mathcal{D}$ are cartesian categories) then a functor $\mathcal{C} \to \mathcal{D}$ which is cartesian and sends J-covering families to K-covering families is a morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$.*
- *If J and K are subcanonical then a geometric morphism $g : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ is of the form $\mathbf{Sh}(f)$ for some f if and only if the inverse image functor g^* sends representables to representables; if this is the case then f is isomorphic to the restriction of g^* to the full subcategories of representables.*

Comorphisms of sites

Recall that a comorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$ is a functor $\pi : \mathcal{D} \to \mathcal{C}$ such that for any $d \in \mathcal{D}$ and any J-covering sieve S on $\pi(d)$ there is a K-covering sieve R on d such that $\pi(R) \subseteq S$.

Proposition

Every comorphism of sites $\pi : \mathcal{D} \to \mathcal{C}$ induces a flat and J-continuous functor $A_\pi : \mathcal{C} \to \mathbf{Sh}(\mathcal{D}, K)$ given by

$$A_\pi(c) = a_K(\mathrm{Hom}_{\mathcal{C}}(\pi(-), c))$$

and hence a geometric morphism

$$f : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

with inverse image $f^*(F) \cong a_K(F \circ \pi)$ for any J-sheaf F on $\mathcal{C}$.

Kan extensions

The direct and image functors of geometric morphisms induced by morphisms or comorphisms of sites can be naturally described in terms of Kan extensions.

Recall that, given a functor $f : \mathcal{C} \to \mathcal{D}$,

- the right Kan extension $\mathrm{Ran}_{f^{\mathrm{op}}}$ along f^{op}, which is right adjoint to the functor $f^* : [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, is given by the following formula:

$$\mathrm{Ran}_{f^{\mathrm{op}}}(F)(b) = \varprojlim_{\phi : fa \to b} F(a),$$

  where the limit is taken over the opposite of the comma category $(f \downarrow b)$.

- The left adjoint to f^* is the left Kan extension $\mathrm{Lan}_{f^{\mathrm{op}}}$ along f^{op}, which is left adjoint to f^*, is given by the following formula:

$$\mathrm{Lan}_{f^{\mathrm{op}}}(F)(b) = \varinjlim_{\phi : b \to fa} F(a),$$

  where the colimit is taken over the opposite of the comma category $(b \downarrow f)$.

Geometric morphisms and Kan extensions

Proposition

(i) *Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of small-generated sites. Then*

- *the direct image $\mathbf{Sh}(F)_*$ of the geometric morphism*

$$\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

  induced by F is given by the restriction to sheaves of F^*;
- *the inverse image $\mathbf{Sh}(F)^*$ of $\mathbf{Sh}(F)$ is given by*

$$a_K \circ \mathrm{Lan}_{F^{\mathrm{op}}} \circ i_J ,$$

  where $\mathrm{Lan}_{F^{\mathrm{op}}}$ is the left Kan extension and i_J is the inclusion $\mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

(ii) *Let $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of small-generated sites. Then*

- *the direct image $(C_F)_*$ of the geometric morphism*

$$C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

  induced by F is given by the restriction to sheaves of the right Kan extension $\mathrm{Ran}_{F^{\mathrm{op}}}$;
- *the inverse image $(C_F)^*$ of C_F is given by*

$$a_J \circ F^* \circ i_K ,$$

  where i_K is the inclusion $\mathbf{Sh}(\mathcal{D}, K) \hookrightarrow [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$.

Unifying morphisms and comorphisms of sites

In order to better contextualize the role of morphisms and of comorphisms of sites, we will now briefly review the philosophy of toposes as 'bridges', which also inspires all the other results presented in this course.

In fact, we shall unify the notions of morphism and comorphisms of sites by interpreting them as two fundamentally different ways of describing morphisms of toposes which correspond to each other under a 'bridge'.

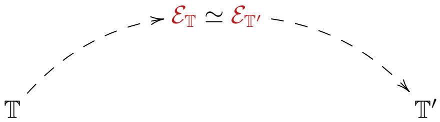
More specifically, morphisms of sites provide an 'algebraic' perspective on morphisms of toposes, while comorphisms of sites provide a 'geometric' perspective on them.

Topos-theoretic invariants

- By a topos-theoretic invariant we mean any notion which is invariant under categorical equivalence of toposes.
- The notion of a geometric morphism of toposes is a fundamental invariant, which has notably allowed to build general comology theories starting from the categories of internal abelian groups or modules in toposes. In particular, the topos-theoretic viewpoint has allowed Grothendieck to refine and enrich the study of cohomology, up to the so-called 'six-operation formalism'. The cohomological invariants have had a tremendous impact on the development of modern Algebraic Geometry and beyond.
- On the other hand, also homotopy-theoretic invariants such as the fundamental group and the higher homotopy groups can be defined as invarants of toposes.
- Still, these are by no means the only invariants that one can consider on toposes: indeed, there are infinitely many invariants of toposes (of algebraic, logical, geometric or whatever nature), the notion of identity for toposes being simply categorical equivalence.

Toposes as *bridges*

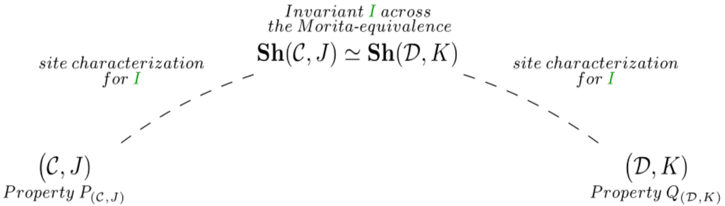
- In the topos-theoretic study of theories or 'concrete' mathematical contexts, the latter are represented by sites (of definition of their classifying topos or of some other topos naturally attached to them).
- Grothendieck toposes can be effectively used as 'bridges' for transferring notions, properties and results across them:



- The transfer of information takes place by expressing topos-theoretic invariants in terms of the different sites of definition (or, more generally, presentations) for the given topos.
- As such, different properties (resp. constructions) arising in the context of the two presentations are seen to be different *manifestations* of a *unique* property (resp. construction) lying at the topos-theoretic level.

The 'bridge' technique

- Decks of 'bridges': Morita-equivalences (that is, equivalences between different presentations of a given topos, or more generally morphisms or other kinds of relations between toposes)
- Arches of 'bridges': Site characterizations for topos-theoretic invariants (or more generally 'unravelings' of topos-theoretic invariants in terms of concrete representations of the relevant topos)



For example, this 'bridge' yields a logical equivalence between the 'concrete' properties $P_{(\mathcal{C},J)}$ and $Q_{(\mathcal{D},K)}$, interpreted in this context as manifestations of a unique property I lying at the level of the topos.

Toposes as *bridges*

- This methodology is technically effective because the relationship between a topos and its representations is often very natural, enabling us to transfer invariants across different representations.
- On the other hand, the 'bridge' technique is highly non-trivial, in the sense that it often yields deep and surprising results. This is due to the fact that a given invariant can manifest itself in significanly different ways in the context of different presentations.
- The level of generality represented by topos-theoretic invariants is ideal to capture several important features of mathematical theories and constructions.

Relating morphisms and comorphisms of sites

The inspiration for our constructions is provided by the following result:

Proposition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites, and $(F : \mathcal{C} \to \mathcal{D} \dashv G : \mathcal{D} \to \mathcal{C})$ adjoint functors. Then

(i) *G is a morphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$ if and only if F is a comorphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$.*

(ii) *In the situation of (i), the geometric morphism C_F induced by F coincides with the geometric morphism $\mathbf{Sh}(G)$ induced by G.*

The key idea is to replace the given sites of definition with Morita-equivalent ones in such a way that every morphism (resp. comorphism) of sites acquires a left (resp. right) adjoint, not necessarily in the classical categorial sense but in the weaker topos-theoretic sense of the associated comma categories having equivalent associated toposes.

From morphisms to comorphisms of sites

We shall turn a morphism of sites into a comorphism of sites by replacing the original codomain site with a site related to it by a morphism inducing an equivalence of toposes such that the composite of the given morphism of sites with it admits a left adjoint; this left adjoint will then be a comorphism of sites inducing the same geometric morphism (by the above proposition).

We shall denote by $(F \downarrow G)$, for two functors $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathcal{C}$, the comma category whose objects are the triplets (a, b, α) where $a \in \mathcal{A}$, $b \in \mathcal{B}$ and α is an arrow $F(a) \to G(b)$ in $\mathcal{C}$ (and whose arrows are defined in the obvious way).

In particular, given a functor $F : \mathcal{C} \to \mathcal{D}$, the objects of $(1_{\mathcal{D}} \downarrow F)$ are triplets of the form $(d, c, \alpha : d \to F(c))$ where $c \in \mathcal{C}$, $d \in \mathcal{D}$ and α is an arrow in $\mathcal{D}$.

From morphisms to comorphisms of sites

Theorem

Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of small-generated sites. Let i_F be the functor $\mathcal{C} \to (1_{\mathcal{D}} \downarrow F)$ sending any object c of $\mathcal{C}$ to the triplet $(F(c), c, 1_{F(c)})$ (and acting on arrows in the obvious way), and $\pi_{\mathcal{C}} : (1_{\mathcal{D}} \downarrow F) \to \mathcal{C}$ and $\pi_{\mathcal{D}} : (1_{\mathcal{D}} \downarrow F) \to \mathcal{D}$ the canonical projection functors. Let $\tilde{K}$ be the Grothendieck topology on $(1_{\mathcal{D}} \downarrow F)$ whose covering sieves are those whose image under $\pi_{\mathcal{D}}$ is K-covering. Then

(i) *$\pi_{\mathcal{C}} \dashv i_F$, $\pi_{\mathcal{D}} \circ i_F = F$, i_F is a morphism of sites $(\mathcal{C}, J) \to ((1_{\mathcal{D}} \downarrow F), \tilde{K})$ and $c_F := \pi_{\mathcal{C}}$ is a comorphism of sites $((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{C}, J)$;*

(ii) *$\pi_{\mathcal{D}} : ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{D}, K)$ is both a morphism of sites and a comorphism of sites inducing equivalences*

$$C_{\pi_{\mathcal{D}}} : \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to \mathbf{Sh}(\mathcal{D}, K)$$

and

$$\mathbf{Sh}(\pi_{\mathcal{D}}) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K})$$

which are quasi-inverse to each other and make the following triangle commute:

$$\begin{array}{ccc} \mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) & \underset{\mathbf{Sh}(\pi_{\mathcal{D}})}{\overset{C_{\pi_{\mathcal{D}}}}{\rightleftarrows}} \;\sim & \mathbf{Sh}(\mathcal{D}, K) \\ & {\scriptstyle C_{\pi_{\mathcal{C}}} \cong \mathbf{Sh}(i_F)} \searrow \quad \swarrow {\scriptstyle \mathbf{Sh}(F)} & \\ & \mathbf{Sh}(\mathcal{C}, J) & \end{array}$$

From comorphisms to morphisms of sites

Below, we shall abbreviate by $\hat{\mathcal{D}}$ the category of presheaves on a small category $\mathcal{D}$.

Theorem

Let $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of small-generated sites. Let $\pi'_{\mathcal{C}} : (F \downarrow 1_{\mathcal{C}}) \to \mathcal{C}$ and $\pi'_{\mathcal{D}} : (F \downarrow 1_{\mathcal{C}}) \to \mathcal{D}$ be the canonical projection functors and $j_F : \mathcal{D} \to (F \downarrow 1_{\mathcal{C}})$ the functor sending any object d of $\mathcal{D}$ to the triplet $(d, F(d), 1_{F(d)})$. Let $\overline{K}$ be the Grothendieck topology on $(F \downarrow 1_{\mathcal{C}})$ whose covering families are those which are sent by $\pi'_{\mathcal{D}}$ to K-covering families. Then

(i) *$j_F \dashv \pi'_{\mathcal{D}}$, $\pi'_{\mathcal{C}} \circ j_F = F$, $\pi'_{\mathcal{C}}$ is a comorphism of sites $(F \downarrow 1_{\mathcal{C}}, \overline{K}) \to (\mathcal{C}, J)$ and j_F is a (full and faithful) comorphism and dense morphism of sites $(\mathcal{D}, K) \to (F \downarrow 1_{\mathcal{C}}, \overline{K})$;*

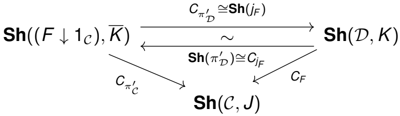
(ii) *$\pi'_{\mathcal{D}}$ is both a morphism and a comorphism of sites $((F \downarrow 1_{\mathcal{C}}), \overline{K}) \to (\mathcal{D}, K)$ inducing equivalences*

$$C_{\pi'_{\mathcal{D}}} : \mathbf{Sh}((F \downarrow 1_{\mathcal{C}}), \overline{K}) \to \mathbf{Sh}(\mathcal{D}, K)$$

and

$$\mathbf{Sh}(\pi'_{\mathcal{D}}) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}((F \downarrow 1_{\mathcal{C}}), \overline{K})$$

which are quasi-inverse to each other and make the following triangle commute:



From comorphisms to morphisms of sites

(iii) With the comorphism of sites $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ we can associate the morphism of sites

$$m_F : (\mathcal{C}, J) \to (\hat{\mathcal{D}}, \hat{K})$$

sending an object c of $\mathcal{C}$ to the presheaf $\mathrm{Hom}_{\mathcal{C}}(F(-), c)$ and $\hat{K}$ is the extension of the Grothendieck topology K along the Yoneda embedding $\mathcal{D} \to \hat{\mathcal{D}}$, which induces a geometric morphism $\mathbf{Sh}(m_F)$ making the following triangle commute:

$$\begin{array}{ccc}
\mathbf{Sh}(\hat{\mathcal{D}}, \hat{K}) & \underset{C_{y_{\mathcal{D}}}}{\overset{\mathbf{Sh}(y_{\mathcal{D}})}{\underset{\sim}{\rightleftarrows}}} & \mathbf{Sh}(\mathcal{D}, K) \\
 & {\scriptstyle \mathbf{Sh}(m_F)} \searrow \quad \swarrow {\scriptstyle C_F} & \\
 & \mathbf{Sh}(\mathcal{C}, J) &
\end{array}$$

Bridging morphisms and comorphisms of sites

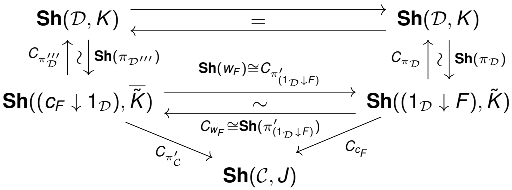
Theorem

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites.

(i) *Let $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of sites, with corresponding comorphism of sites $c_F : ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{C}, J)$ as above. Let $\pi_{\mathcal{D}} : ((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to (\mathcal{D}, K)$ be the canonical projection functor, and let*

$$w_F : (1_{\mathcal{D}} \downarrow F) \to (c_F \downarrow 1_{\mathcal{D}})$$

be the functor j_{c_F}, sending an object A of $(1_{\mathcal{D}} \downarrow F)$ to the object $(A, c_F(A), 1_{c_F(A)} : c_F(A) \to c_F(A))$. Then w_F is both a (full and faithful) comorphism and a dense morphism of sites $((1_{\mathcal{D}} \downarrow F), \tilde{K}) \to ((c_F \downarrow 1_{\mathcal{D}}), \overline{\tilde{K}})$ satisfying the relation $\pi'''_{\mathcal{D}} \circ w_F = \pi_{\mathcal{D}}$ and inducing an equivalence relating F and c_F, which makes the following diagram commute (where $\pi'''_{\mathcal{D}}$ denotes the canonical projection functor $(c_F \downarrow 1_{\mathcal{D}}) \to \mathcal{D}$):



Bridging morphisms and comorphisms of sites

(ii) Let $G : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of sites, with corresponding morphism of sites $m_G : (\mathcal{C}, J) \to (\hat{\mathcal{D}}, \hat{K})$ as above. Let

$$z_G : (G \downarrow 1_{\mathcal{C}}) \to (1_{\hat{\mathcal{D}}} \downarrow m_G)$$

be the functor sending any object $(d, c, \alpha : G(d) \to c)$ of $(G \downarrow 1_{\mathcal{C}})$ to the object $(y_{\mathcal{D}}(d), c, \overline{\alpha} : y_{\mathcal{D}}(d) \to m_G(c))$ of $(1_{\hat{\mathcal{D}}} \downarrow m_G)$, where $\overline{\alpha}$ is the arrow corresponding to the element α of m_G via the Yoneda Lemma. Then z_G is both a (full and faithful) comorphism and a dense morphism of sites $((G \downarrow 1_{\mathcal{C}}), \overline{K}) \to ((1_{\hat{\mathcal{D}}} \downarrow m_G), \widetilde{\hat{K}})$ satisfying the relation $\pi_{\hat{\mathcal{D}}} \circ z_G = y_{\mathcal{D}} \circ \pi'_{\mathcal{D}}$ and inducing an equivalence relating G and m_G, which makes the following diagram commute:

$$\begin{array}{ccc}
\mathbf{Sh}(\hat{\mathcal{D}}, \hat{K}) & \underset{C_{y_{\mathcal{D}}}}{\overset{\mathbf{Sh}(y_{\mathcal{D}})}{\rightleftarrows}} \; \sim & \mathbf{Sh}(\mathcal{D}, K) \\
C_{\pi_{\hat{\mathcal{D}}}} \uparrow \wr \downarrow \mathbf{Sh}(\pi_{\hat{\mathcal{D}}}) & & \mathbf{Sh}(j_G) \cong C_{\pi_{\mathcal{D}'}} \uparrow \wr \downarrow \mathbf{Sh}(\pi'_{\mathcal{D}}) \cong C_{j_G} \\
\mathbf{Sh}((1_{\hat{\mathcal{D}}} \downarrow m_G), \widetilde{\hat{K}}) & \underset{C_{z_G}}{\overset{\mathbf{Sh}(z_G)}{\rightleftarrows}} \; \sim & \mathbf{Sh}((G \downarrow 1_{\mathcal{C}}), \overline{K}) \\
& {\scriptstyle C_{\pi_{\mathcal{C}}} \cong \mathbf{Sh}(i_{m_G})} \searrow \quad \swarrow {\scriptstyle C_{\pi'_{\mathcal{C}}}} & \\
& \mathbf{Sh}(\mathcal{C}, J) &
\end{array}$$

Bridging morphisms and comorphisms of sites

We shall call a functor which both a morphism and a comorphism of sites a bimorphism of sites.

The above theorem shows that the relationship between a morphism F (resp. comorphism G) of sites and the associated comorphism c_F (resp. morphism m_F) of sites is captured by the equivalence

$$\mathbf{Sh}((1_{\mathcal{D}} \downarrow F), \tilde{K}) \simeq \mathbf{Sh}((c_F \downarrow 1_{\mathcal{D}}), \overline{\tilde{K}})$$

(resp.

$$\mathbf{Sh}((G \downarrow 1_{\mathcal{C}}), \overline{K}) \simeq \mathbf{Sh}((1_{\hat{\mathcal{D}}} \downarrow m_G), \widetilde{\hat{K}}))$$

of toposes over $\mathbf{Sh}(\mathcal{C}, J)$ induced by the bimorphism of sites w_F (resp. z_G) over $\mathcal{C}$.

Our theorem then tells us that F and c_F (resp. G and m_G) are not adjoint to each other in a concrete sense (that is, at the level of sites), since they are not defined between a pair of categories, nor the categories $(1_{\mathcal{D}} \downarrow F)$ and $(c_F \downarrow 1_{\mathcal{D}})$ (resp. the categories $(G \downarrow 1_{\mathcal{C}})$ and $(1_{\hat{\mathcal{D}}} \downarrow m_G)$) are equivalent in general; nonetheless, they become 'abstractly' adjoint in the world of toposes since toposes naturally attached to such categories are equivalent.

The dual adjunction

Definition

Let $(\mathcal{C}, J)$ be a small-generated site.

(a) The category $\mathbf{Mor}_{(\mathcal{C},J)}$ has as objects the morphisms of sites from $(\mathcal{C}, J)$ to a small generated site $(\mathcal{D}, K)$ and as arrows

$$(F : (\mathcal{C}, J) \to (\mathcal{D}, K)) \to (F' : (\mathcal{C}, J) \to (\mathcal{D}', K'))$$

between any two such morphisms the geometric morphisms

$$f : \mathbf{Sh}(\mathcal{D}', K') \to \mathbf{Sh}(\mathcal{D}, K)$$

such that $\mathbf{Sh}(F) \circ f \cong \mathbf{Sh}(F')$:

$$\begin{array}{ccc} \mathbf{Sh}(\mathcal{C}, J) & \xleftarrow{\mathbf{Sh}(F)} & \mathbf{Sh}(\mathcal{D}, K) \\ & {\scriptstyle \mathbf{Sh}(F')}\nwarrow & \uparrow{\scriptstyle f} \\ & & \mathbf{Sh}(\mathcal{D}', K') \end{array}$$

(b) The category $\mathbf{Com}_{(\mathcal{C},J)}$ has as objects the comorphisms of sites from a small-generated site $(\mathcal{D}, K)$ to $(\mathcal{C}, J)$ and as arrows

$$(U : (\mathcal{D}, K) \to (\mathcal{C}, J)) \to (U' : (\mathcal{D}', K') \to (\mathcal{C}, J))$$

between any two such comorphisms the geometric morphisms

$$g : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{D}', K')$$

such that $C_{U'} \circ g \cong C_U$:

$$\begin{array}{ccc} \mathbf{Sh}(\mathcal{D}, K) & \xrightarrow{C_U} & \mathbf{Sh}(\mathcal{C}, J) \\ \downarrow{\scriptstyle g} & \nearrow{\scriptstyle C_{U'}} & \\ \mathbf{Sh}(\mathcal{D}', K') & & \end{array}$$

The dual adjunction

The assignments $F \mapsto c_F$ and $G \mapsto m_G$ introduced above naturally define two functors

$$C : (\mathbf{Mor}_{(\mathcal{C},J)})^{\mathrm{op}} \to \mathbf{Com}_{(\mathcal{C},J)}$$

and

$$M : \mathbf{Com}_{(\mathcal{C},J)} \to (\mathbf{Mor}_{(\mathcal{C},J)})^{\mathrm{op}} .$$

Theorem

The functors

$$C : (\mathbf{Mor}_{(\mathcal{C},J)})^{\mathrm{op}} \to \mathbf{Com}_{(\mathcal{C},J)}$$

and

$$M : \mathbf{Com}_{(\mathcal{C},J)} \to (\mathbf{Mor}_{(\mathcal{C},J)})^{\mathrm{op}}$$

are (2-categorically) adjoint (C on the right and M on the left) and quasi-inverse to each other.

The canonical stack of a geometric morphism

Corollary

Let $f : \mathcal{F} \to \mathcal{E}$ be a geometric morphism. Then the canonical projection functor

$$\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$$

is a comorphism of sites $((1_{\mathcal{F}} \downarrow f^*), \widetilde{J_{\mathcal{F}}^{\text{can}}}) \to (\mathcal{E}, J_{\mathcal{E}}^{\text{can}})$ such that $f = C_{\pi_{\mathcal{E}}}$.

As we shall see below, the functor $\pi_{\mathcal{E}} : (1_{\mathcal{F}} \downarrow f^*) \to \mathcal{E}$ is actually a stack on $\mathcal{E}$, which we call the canonical stack of f: from an indexed point of view, this stack sends any object E of $\mathcal{E}$ to the topos $\mathcal{F}/f^*(E)$ and any arrow $u : E' \to E$ to the pullback functor $u^* : \mathcal{F}/f^*(E) \to \mathcal{F}/f^*(E')$.

We shall call the Grothendieck topology $\widetilde{J_{\mathcal{F}}^{\text{can}}}$ on $(1_{\mathcal{F}} \downarrow f^*)$ the relative topology of f.

By taking f to be the identity, and choosing a site of definition $(\mathcal{C}, J)$ for $\mathcal{E}$, we obtain the canonical stack $\mathcal{S}_{(\mathcal{C},J)}$ on $(\mathcal{C}, J)$, which sends any object c of $\mathcal{C}$ to the topos $\mathbf{Sh}(\mathcal{C}, J)/l(c)$. The above corollary thus specializes to an equivalence

$$\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{S}_{(\mathcal{C},J)}, \widetilde{J^{\text{can}}_{\mathbf{Sh}(\mathcal{C},J)}}),$$

which represents a 'thickening' of the usual representation of a Grothendieck topos as the topos of sheaves over itself with respect to the canonical topology.

From comorphisms of sites to fibrations

The following result shows that one can naturally associate with a comorphism of sites a fibration inducing the same geometric morphism.

Definition

The fibration of generalized elements of a functor $F : \mathcal{D} \to \mathcal{C}$ is the canonical projection functor $\pi^F_{\mathcal{C}} : (1_{\mathcal{C}} \downarrow F) \to \mathcal{C}$.

Theorem

Let $F : (\mathcal{D}, K) \to (\mathcal{C}, J)$ be a comorphism of small-generated sites, i'_F the canonical functor $\mathcal{D} \to (1_{\mathcal{C}} \downarrow F)$ and $K^{i'_F}$ the Grothendieck topology on $(1_{\mathcal{C}} \downarrow F)$ whose covering sieves are those whose pullback along any arrow whose domain is an object of the form $i'_F(d)$ contains the image under i'_F of a K-covering sieve on d. Let $\pi^F_{\mathcal{C}}$ and $\pi^F_{\mathcal{D}}$ be the canonical projections from $(1_{\mathcal{C}} \downarrow F)$ respectively to $\mathcal{C}$ and $\mathcal{D}$. Then

(i) $\pi^F_{\mathcal{D}} \dashv i'_F$, $\pi^F_{\mathcal{C}} \circ i'_F = F$, $\pi^F_{\mathcal{C}}$ *is a comorphism of sites* $((1_{\mathcal{C}} \downarrow F), K^{i'_F}) \to (\mathcal{C}, J)$ *and* $\pi^F_{\mathcal{D}}$ *is a comorphism of sites* $((1_{\mathcal{C}} \downarrow F), K^{i'_F}) \to (\mathcal{D}, K)$;

(ii) i'_F *is both a (full and faithful) comorphism of sites and a dense morphism of sites* $(\mathcal{D}, K) \to ((1_{\mathcal{C}} \downarrow F), K^{i'_F})$ *inducing equivalences*

$$C_{i'_F} : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}((1_{\mathcal{C}} \downarrow F), K^{i'_F})$$

and

$$\mathbf{Sh}(i'_F) : \mathbf{Sh}((1_{\mathcal{C}} \downarrow F), K^{i'_F}) \to \mathbf{Sh}(\mathcal{D}, K)$$

which are quasi-inverse to each other and make the following triangle commute:

$$\begin{array}{ccc} \mathbf{Sh}((1_{\mathcal{C}} \downarrow F), K^{i'_F}) & \underset{C_{i'_F}}{\overset{\mathbf{Sh}(i'_F) \cong C_{\pi^F_{\mathcal{D}}}}{\rightleftarrows}} \; \sim & \mathbf{Sh}(\mathcal{D}, K) \\ & {\scriptstyle C_{\pi^F_{\mathcal{C}}}} \searrow \quad \swarrow {\scriptstyle C_F} & \\ & \mathbf{Sh}(\mathcal{C}, J) & \end{array}$$

Fibrations as comorphisms of sites

In the converse direction, every fibration can be naturally regarded as a comorphism of sites, as follows.

Recall that, given a functor $A : \mathcal{C} \to \mathcal{D}$ and a Grothendieck topology K in $\mathcal{D}$, there is a smallest Grothendieck topology on $\mathcal{C}$, which makes A a comorphism of sites to $(\mathcal{D}, K)$. This topology, which we denote by M_K^A, is generated by the (pullback-stable) family of sieves of the form $S_R^A := \{f : \mathrm{dom}(f) \to c \mid A(f) \in R\}$ for an object c of $\mathcal{C}$ and a K-covering sieve R on $A(c)$.

Proposition

If A is a fibration, the topology M_K^A admits the following simpler description: a sieve R is M_K^A-covering if and only if the collection of cartesian arrows in R is sent by A to a K-covering family.

We shall call M_K^A the Giraud topology induced by K, in honour of Jean Giraud, who used it for constructing the classifying topos $\mathbf{Sh}(\mathcal{C}, M_K^A)$ of a stack A on $(\mathcal{D}, K)$.

Proposition

For any Grothendieck topology K on $\mathcal{D}$, every morphism of fibrations $(A : \mathcal{C} \to \mathcal{D}) \to (A' : \mathcal{C}' \to \mathcal{D})$ yields a comorphism of sites $(\mathcal{C}, M_K^A) \to (\mathcal{C}', M_K^{A'})$.

Weak morphisms of toposes

Definition

A *weak morphism of toposes* $f : \mathcal{E} \to \mathcal{F}$ is a pair of adjoint functors $(f^* \dashv f_*)$.

As in the case of geometric morphism, we call f_* the *direct image* of f and f^* the *inverse image* of f.

Proposition

Let $i : \mathcal{F} \hookrightarrow \mathcal{E}$ be the geometric inclusion of a subtopos $\mathcal{F}$ of a Grothendieck topos $\mathcal{E}$ into $\mathcal{E}$, and let $f : \mathcal{G} \to \mathcal{E}$ be a weak morphism from a Grothendieck topos $\mathcal{G}$. Then the following conditions are equivalent:

(i) *The weak morphism f factors through i;*
(ii) *The direct image f_* takes values in $\mathcal{F}$ (that is, factors through i_*);*
(iii) *The inverse image f^* factors (necessarily uniquely up to isomorphism) through i^*.*

Corollary

Let $A : \mathcal{C} \to \mathcal{E}$ be a functor from an essentially small category $\mathcal{C}$ to a Grothendieck topos $\mathcal{E}$, and J be a Grothendieck topology on $\mathcal{C}$. Then the following conditions are equivalent:

(i) *The weak morphism $(L_A \dashv R_A)$ factors through the canonical geometric inclusion $i : \mathbf{Sh}(\mathcal{C}, J) \hookrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}]$;*
(ii) *The functor R_A takes values in $\mathbf{Sh}(\mathcal{C}, J)$;*
(iii) *The functor L_A factors (necessarily uniquely up to isomorphism) through the associated sheaf functor $a_J : [\mathcal{C}^{\text{op}}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$.*

Continuous functors

The above result motivates the following definition:

Definition

- (a) Given a small-generated site $(\mathcal{C}, J)$, we say that a functor $A : \mathcal{C} \to \mathcal{E}$ is J-continuous if the hom functor $R_A : \mathcal{E} \to [\mathcal{C}^{op}, \mathbf{Set}]$ takes values into $\mathbf{Sh}(\mathcal{C}, J)$ (equivalently, if the functor $L_A : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathcal{E}$ factors through $a_J : [\mathcal{C}^{op}, \mathbf{Set}] \to \mathbf{Sh}(\mathcal{C}, J)$).
- (b) Given small-generated sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, a functor $A : \mathcal{C} \to \mathcal{D}$ is said to be (J, K)-continuous if $l' \circ A$ is J-continuous, where l' is the canonical functor $\mathcal{D} \to \mathbf{Sh}(\mathcal{D}, K)$.

The following proposition shows that the above definition is equivalent to Grothendieck's notion of continuous functor:

Proposition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites and $A : \mathcal{C} \to \mathcal{D}$ a functor. Then the following conditions are equivalent:

- (i) *A is (J, K)-continuous.*
- (ii) *The functor*

$$D_A := (- \circ A^{op}) : [\mathcal{D}^{op}, \mathbf{Set}] \to [\mathcal{C}^{op}, \mathbf{Set}]$$

restricts to a functor $\mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$.

Classifying weak morphisms of toposes

Let $[\mathcal{C}, \mathcal{E}]_J$ be the full subcategory of $[\mathcal{C}, \mathcal{E}]$ on the J-continuous functors.

Proposition

Let $\mathcal{C}$ a locally small category and $\mathcal{E}$ a Grothendieck topos.

(i) *There is an equivalence*

$$\mathbf{Wmor}(\mathcal{E}, [\mathcal{C}^{\text{op}}, \mathbf{Set}]) \simeq [\mathcal{C}, \mathcal{E}]$$

sending a weak morphism $f = (f^* \dashv f_*)$ to the functor $f^* \circ y_{\mathcal{C}}$.

(ii) *For any Grothendieck topology J on $\mathcal{C}$ making $(\mathcal{C}, J)$ a small-generated site, the above equivalence restricts to an equivalence*

$$\mathbf{Wmor}(\mathcal{E}, \mathbf{Sh}(\mathcal{C}, J)) \simeq [\mathcal{C}, \mathcal{E}]_J$$

sending a weak morphism $g = (g^* \dashv g_*)$ to the functor $g^* \circ l$.

Weak morphisms of sites

These results motivate the following

Definition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a weak morphism of sites if it is (J, K)-continuous.

Note that this notion generalizes that of morphism of sites; indeed, as morphisms of sites induce geometric morphisms of toposes, so weak morphisms of sites induce weak morphisms of toposes:

Proposition

Any weak morphism $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ of small-generated sites induces a weak geometric morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle l'} \\ \mathbf{Sh}(\mathcal{C}, J) & \xrightarrow{\mathbf{Sh}(F)^*} & \mathbf{Sh}(\mathcal{D}, K) \end{array}$$

Conversely, any weak geometric morphism $f = (f^* \dashv f_*)$ such that $f^* \circ l$ factors through l' is induced by a (necessarily unique, if K is subcanonical) weak morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$.

Continuous functors

Proposition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites and $\mathcal{E}$ a Grothendieck topos. Then

1. *A functor $A : \mathcal{C} \to \mathcal{E}$ is J-continuous if and only if for any J-covering sieve S on an object c*

$$A(c) = \varinjlim_{f:d\to c\in S} A(d)$$

for each J-covering sieve S on an object c (where the colimit is indexed by the category $\int S$ of elements of S).

2. *A functor $A : \mathcal{C} \to \mathcal{D}$ is (J, K)-continuous if and only if for any J-covering sieve S on an object c the canonical cocone with vertex $A(c)$ on the diagram $\{A(\mathrm{dom}(f)) \mid f \in S\}$ indexed over $\int S$ is sent by l' to a colimit in the topos $\mathbf{Sh}(\mathcal{D}, K)$.*

3. *Every J-continuous functor $A : \mathcal{C} \to \mathcal{E}$ is J-continuous in the sense of Mac Lane and Moerdijk (that is, sends J-covering families to epimorphic families), and the converse is true if A is flat (but not in general). More generally, every (J, K)-continuous functor $(\mathcal{C}, J) \to (\mathcal{D}, K)$ is cover-preserving, and every morphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$ is (J, K)-continuous.*

Continuity and cofinaliy

The above proposition suggests that the property of J-continuity could be interpreted as a sort of cofinality condition.

Indeed, if A is J-continuous then in particular A sends any J-covering sieve S on an object c of $\mathcal{C}$ to an epimorphic family and hence $A(c)$ is the colimit of the cocone under the diagram whose vertices are the objects of the form $A(d)$ where d is the domain of an arrow $f : d \to c$ in S and whose arrows are all the arrows in $\mathcal{E}$ over $A(c)$ between such objects.

So the condition for A to be J-continuous amounts precisely to the assertion that A sends S to an epimorphic family and that this colimit be equal to the colimit $\varinjlim_{f:d \to c \in S} A(d)$.

In order to formally express continuity as a form of cofinality, we are going to introduce relative cofinality conditions.

Relative cofinality

Proposition

Let $(\mathcal{C}, J)$ be a small-generated site and $F : \mathcal{A} \to \mathcal{C}$ and $F' : \mathcal{A}' \to \mathcal{C}$ two functors to $\mathcal{C}$ related by a functor $\xi : \mathcal{A} \to \mathcal{A}'$ and a natural transformation $\alpha : F \to F' \circ \xi$. Let R_c (resp. R'_c), for any $c \in \mathcal{C}$, be the equivalence relations on the objects of the category $(c \downarrow F)$ (resp. of $(c \downarrow F')$) given by the relation of belonging to the same connected component.
Then the canonical arrow

$$\tilde{\alpha} : \mathrm{colim}_{[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}(y_{\mathcal{C}} \circ F) \to \mathrm{colim}_{[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]}(y_{\mathcal{C}} \circ F')$$

is sent by a_J to an isomorphism

$$a_J(\tilde{\alpha}) : \mathrm{colim}_{\mathbf{Sh}(\mathcal{C},J)}(l \circ F) \to \mathrm{colim}_{\mathbf{Sh}(\mathcal{C},J)}(l \circ F)$$

if and only if the pair (ξ, α) satisfies the following 'relative cofinality' conditions:

1. *For any object c of $\mathcal{C}$ and any arrow $y : c \to F'(a')$ in $\mathcal{C}$ there are a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ and for each $i \in I$ an object a_i of $\mathcal{A}$ and an arrow $y_i : c_i \to F(a_i)$ such that $(y \circ f_i, \alpha(a_i) \circ y_i) \in R'_{c_i}$.*
2. *For any object c of $\mathcal{C}$ and any arrows $x : c \to F(a)$ and $x' : c \to F(b)$ in $\mathcal{C}$ such that $(\alpha(a) \circ x, \alpha(b) \circ x') \in R'_c$ there is a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ such that $(x \circ f_i, x' \circ f_i) \in R_{c_i}$ for each $i \in I$.*

J-cofinal functors

It is interesting to apply the proposition in two notable particular cases:

1. $F = \xi : \mathcal{A} \to \mathcal{C}$, $F' = 1_{\mathcal{C}}$, α is the identity.
2. F' is the forgetful functor $U_{c_0} : \mathcal{C}/c_0 \to \mathcal{C}$ for an object c_0 of $\mathcal{C}$, ξ is a cocone $\{\xi_a : F(a) \to c_0 \mid a \in \mathcal{A}\}$ under the functor F with vertex c_0 and α is the identity.

Formulating the thesis of the proposition in these particular cases leads us to introduce the following

Definition

Given a small-generated site $(\mathcal{C}, J)$, a functor $F : \mathcal{A} \to \mathcal{C}$ is said to be *J-cofinal* if the following conditions are satisfied:

(i) For any object c of $\mathcal{C}$ there are a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ and for each $i \in I$ an object a_i of $\mathcal{A}$ and an arrow $y_i : c_i \to F(a_i)$.

(ii) For any object c of $\mathcal{C}$ and any arrows $x : c \to F(a)$ and $x' : c \to F(b)$ in $\mathcal{C}$ there is a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ such that $x \circ f_i$ and $x' \circ f_i$ belong to the same connected component of the category $(c_i \downarrow F)$ for each $i \in I$.

Two corollaries

The proposition thus yields the following two results:

Corollary

Let $(\mathcal{C}, J)$ be a small-generated site and $F : \mathcal{A} \to \mathcal{C}$ a functor. Then F is J-cofinal if and only if the canonical arrow

$$\mathrm{colim}_{\mathbf{Sh}(\mathcal{C},J)}(l \circ F) \to 1_{\mathbf{Sh}(\mathcal{C},J)}$$

is an isomorphism.

Corollary

Let $D : \mathcal{A} \to \mathcal{C}$ be a functor and ξ a cocone $\{\xi_a : D(a) \to c_0 \mid a \in \mathcal{A}\}$ under D with vertex c_0. Let U_{c_0} be the forgetful functor $\mathcal{C}/c_0 \to \mathcal{C}$, J_{c_0} the Grothendieck topology on $\mathcal{C}/c_0$ whose covering sieves are precisely those whose image under U_{c_0} is J-covering and $D_\xi : \mathcal{A} \to \mathcal{C}/c_0$ the canonical lift of D to $\mathcal{C}/c_0$ (which satisfies $U_{c_0} \circ D_\xi = D$).
Then ξ is sent by the canonical functor $l : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}, J)$ to a colimit cocone if and only if the functor D_ξ is J_{c_0}-cofinal, equivalently if and only if the following conditions are satisfied:

1. *For any object c of $\mathcal{C}$ and any arrow $y : c \to c_0$ in $\mathcal{C}$ there are a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ and for each $i \in I$ an object a_i of $\mathcal{A}$ and an arrow $y_i : c_i \to D(a_i)$ such that $y \circ f_i = \xi_{a_i} \circ y_i$.*
2. *For any object c of $\mathcal{C}$ and any arrows $x : c \to D(a)$ and $x' : c \to D(b)$ in $\mathcal{C}$ such that $\xi_a \circ x = \xi_b \circ x'$ there is a J-covering family $\{f_i : c_i \to c \mid i \in I\}$ such that $x \circ f_i$ and $x' \circ f_i$ belong to the same connected component of the category $(c_i \downarrow D)$ for each $i \in I$.*

Characterization of colimits in toposes

This notion of relative cofinality has several applications. A basic one is the characterization of colimits in Grothendieck toposes in terms of generalized elements:

Corollary

Let $D : \mathcal{A} \to \mathcal{E}$ be a functor from a small category $\mathcal{A}$ to a Grothendieck topos $\mathcal{E}$ and ξ a cocone $\{\xi_a : D(a) \to e_0 \mid a \in \mathcal{A}\}$ under D with vertex e_0. Then ξ is a colimit cocone if and only if the functor D_ξ is $(J^{\text{can}}_{\mathcal{E}})_{e_0}$-cofinal, equivalently if and only if the following conditions are satisfied:

1. *For any object e of $\mathcal{E}$ and any arrow $y : e \to e_0$ in $\mathcal{E}$ there are an epimorphc family $\{f_i : e_i \to e \mid i \in I\}$ in $\mathcal{E}$ and for each $i \in I$ an object a_i of $\mathcal{A}$ and an arrow $y_i : e_i \to D(a_i)$ such that $y \circ f_i = \xi_{a_i} \circ y_i$.*
2. *For any object e of $\mathcal{E}$ and any arrows $x : e \to D(a)$ and $x' : e \to D(b)$ in $\mathcal{E}$ such that $\xi_a \circ x = \xi_b \circ x'$ there is an epimorphic family $\{f_i : e_i \to e \mid i \in I\}$ in $\mathcal{E}$ such that $x \circ f_i$ and $x' \circ f_i$ belong to the same connected component of the category $(e_i \downarrow D)$ for each $i \in I$.*

Characterization of continuous functors

Let

$$D_S^A : \int S \to \mathcal{D}$$

be the functor sending any (d, f) of $\int S$ to $A(d)$, together with the cocone ξ_A with vertex $A(c)$ under it (whose legs are the arrows $A(f) : A(d) = D_S^A((d, f)) \to A(c)$ for any object (d, f) of $\int S$).

Applying one of the above corollaries to it, we obtain the following explicit characterization of (J, K)-continuous functors:

Proposition

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites. Then a functor $A : \mathcal{C} \to \mathcal{D}$ is (J, K)-continuous if and only if it is cover-preserving (i.e., sends J-covering families to K-covering ones) and for any J-covering sieve S on an object c and any commutative square of the form

$$\begin{array}{ccc} d & \longrightarrow & A(c') \\ \downarrow & & \downarrow{\scriptstyle A(f)} \\ A(c'') & \xrightarrow{A(g)} & A(c), \end{array}$$

where $f : c' \to c$ and $g : c'' \to c$ are arbitrary arrows of S, there is a K-covering family $\{d_i \to d \mid i \in I\}$ such that for each $i \in I$, the composites $d_i \to A(c')$ and $d_i \to A(c'')$ belong to the same connected component of the category $(d_i \downarrow D_S^A)$.

Indeed, the conditions of the proposition are equivalent to the requirement that that the lift

$$(D_A^S)_{\xi_A} : \int S \to \mathcal{D}/A(c)$$

of the diagram D_A^S to $\mathcal{D}/A(c)$ induced by the cocone ξ_A be $K_{A(c)}$-cofinal.

Continuity of (morphisms of) fibrations

By using the above characterization of continuous functors, one can prove

Proposition

Let $A : \mathcal{C} \to \mathcal{D}$ be a fibration. Then, for any Grothendieck topology K on $\mathcal{D}$, A is a continuous comorphism of sites $(\mathcal{C}, M_K^A) \to (\mathcal{D}, K)$.

More generally, we have the following result:

Theorem

For any Grothendieck topology K on $\mathcal{D}$, every morphism of fibrations $(A : \mathcal{C} \to \mathcal{D}) \to (A' : \mathcal{C}' \to \mathcal{D})$ is a continuous comorphism of sites $(\mathcal{C}, M_K^A) \to (\mathcal{C}', M_K^{A'})$.

Classifying essential morphisms

Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be essential if its inverse image f^* has a left adjoint, denoted by $f_!$ and called its essential image.

Theorem

Let $(\mathcal{C}, J)$ be a small-generated site, $\mathcal{E}$ a Grothendieck topos. Let $\mathbf{Geom}_{\mathrm{ess}}(\mathbf{Sh}(\mathcal{C}, J), \mathcal{E})$ be the category of essential geometric morphisms, and $\mathbf{Com}_{\mathrm{cont}}((\mathcal{C}, J), (\mathcal{E}, J^{\mathrm{can}}_{\mathcal{E}}))$ the category of J-continuous comorphisms of sites $(\mathcal{C}, J) \to (\mathcal{E}, J^{\mathrm{can}}_{\mathcal{E}})$. Then we have an equivalence

$$\mathbf{Geom}_{\mathrm{ess}}(\mathbf{Sh}(\mathcal{C}, J), \mathcal{E}) \simeq \mathbf{Com}_{\mathrm{cont}}((\mathcal{C}, J), (\mathcal{E}, J^{\mathrm{can}}_{\mathcal{E}}))$$

sending an essential geometric morphism $f = (f_! \dashv f^* \dashv f_*)$ to the comorphism of sites $f_! \circ l$ and a J-continuous comorphism of sites A to the geometric morphism C_A induced by it.

We say that two comorphisms of sites $A, A' : (\mathcal{C}, J) \to (\mathcal{D}, K)$ are K-equivalent if the geometric morphisms C_A and $C_{A'}$ that they induce are isomorphic.

Corollary

Let $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ be small-generated sites. Then we have an equivalence between the essential geometric morphisms $f : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ such that $f_! \circ l : \mathcal{C} \to \mathbf{Sh}(\mathcal{D}, K)$ factors through the canonical functor $l' : \mathcal{D} \to \mathbf{Sh}(\mathcal{D}, K)$ and the (J, K)-continuous comorphism of sites $(\mathcal{C}, J) \to (\mathcal{D}, K)$, considered up to K-equivalence.

Continuous comorphisms of sites

The following result provides alternative characterizations for the property of a comorphism of sites to be continuous:

Proposition

Let $A : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a comorphism of sites. Then the following conditions are equivalent:

1. *A is (J, K)-continuous.*
2. *The left Kan extension functor $\mathrm{Lan}_{A^{\mathrm{op}}} : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ along A^{op} satisfies the property that $a_K \circ \mathrm{Lan}_{A^{\mathrm{op}}}$ factors (necessarily uniquely) through a_J.*
3. *The geometric morphism C_A induced by A is essential and its essential image $(C_A)_!$ makes the following diagram commute:*

$$\begin{array}{ccc} [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] & \xrightarrow{\mathrm{Lan}_{A^{\mathrm{op}}}} & [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \\ \downarrow a_J & & \downarrow a_K \\ \mathbf{Sh}(\mathcal{C}, J) & \xrightarrow{(C_A)_!} & \mathbf{Sh}(\mathcal{D}, K) \end{array}$$

If A induces an essential geometric morphism C_A then there is a canonical morphism $(C_A)_! \circ l \to l' \circ A$, and A is (J, K)-continuous if and only if this morphism is an isomorphism, equivalently if and only if the canonical morphism

$$(C_A)_! \circ a_J \to a_K \circ \mathrm{Lan}_{A^{\mathrm{op}}}$$

is an isomorphism.

Local connectedness

The notion of locally connected morphism represents a natural strengthening of the notion of essential morphism. Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be locally connected if f^* has an $\mathcal{E}$-indexed left adjoint, equivalently for any arrow $h : A \to B$ in $\mathcal{E}$, the square

$$\begin{array}{ccc} \mathcal{F}/f^*(B) & \xrightarrow{(f/B)_!} & \mathcal{E}/B \\ \Big\downarrow{\scriptstyle (f^*(h))^*} & & \Big\downarrow{\scriptstyle h^*} \\ \mathcal{F}/f^*(A) & \xrightarrow{(f/A)_!} & \mathcal{E}/A \end{array}$$

commutes.

The continuity of (morphisms of) fibrations implies that such comorphisms always induce essential geometric morphisms. One might thus wonder if these morphisms always induce locally connected morphisms. Interestingly, one can prove that this is true for fibrations but *not* in general for morphisms of fibrations.

Indexed categories and fibrations

The language in which we shall work for developing relative topos theory is that of indexed categories and fibrations.

- Given a category $\mathcal{C}$, we shall denote by $\mathbf{Ind}_{\mathcal{C}}$ the 2-category of $\mathcal{C}$-indexed categories: it is the 2-category $[\mathcal{C}^{\text{op}}, \mathbf{Cat}]_{ps}$ whose 0-cells are the pseudofunctors $\mathcal{C}^{\text{op}} \to \mathbf{Cat}$, whose 1-cells are the pseudonatural transformations and whose 2-cells are the modifications between them.
- Given a category $\mathcal{C}$, we shall denote by $\mathbf{Fib}_{\mathcal{C}}$ the 2-category of fibrations over $\mathcal{C}$: it is the sub-2-category of $\mathbf{CAT}/\mathcal{C}$ whose 0-cells are the (Street) fibrations $p : \mathcal{D} \to \mathcal{C}$, whose 1-cells are the morphisms of fibrations (with a 'commuting' isomorphism) and whose 2-cells are the natural transformations between them.

  We shall denote by $\mathbf{cFib}_{\mathcal{C}}$ the full sub-2-category of cloven fibrations (i.e. fibrations equipped with a cleavage).

It is well-known that indexed categories and fibrations are in equivalence with each other:

Theorem

For any category $\mathcal{C}$, there is an equivalence of 2-categories between $\mathbf{Ind}_{\mathcal{C}}$ and $\mathbf{cFib}_{\mathcal{C}}$, one half of which is given by the Grothendieck construction and whose other half is given by the functor taking the fibers at the objects of $\mathcal{C}$.

The notion of stack

Definition

Consider a site $(\mathcal{C}, J)$ and a fibration $p : \mathcal{D} \to \mathcal{C}$: then p is a J-prestack (resp. J-stack) if for every J-sieve $m_S : S \rightarrowtail y_{\mathcal{C}}(X)$ the functor

$$- \circ \int m_S : \mathbf{Fib}_{\mathcal{C}}(\mathcal{C}/X, \mathcal{D}) \to \mathbf{Fib}_{\mathcal{C}}(\int S, \mathcal{D})$$

is full and faithful (resp. an equivalence).

Stacks over a site $(\mathcal{C}, J)$ form a 2-full and faithful subcategory of $\mathbf{Ind}_{\mathcal{C}}$, which we will denote by $\mathbf{St}(\mathcal{C}, J)$.

The notion of stack on a site is a higher-categorical generalization of that of sheaf on that site:

Proposition

Consider a site $(\mathcal{C}, J)$ and a presheaf $P : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$: then P is J-separated (resp. J-sheaf) if and only if the fibration $\int P \to \mathcal{C}$ is a J-prestack (resp. J-stack).

We can rewrite the condition for a pseudofunctor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ to be a J-prestack (resp. J-stack) in the language of indexed categories, as the requirement that for every sieve $m_S : S \rightarrowtail y_{\mathcal{C}}(X)$ the functor

$$\mathbf{Ind}_{\mathcal{C}}(y_{\mathcal{C}}(X), \mathbb{D}) \xrightarrow{- \circ m_S} \mathbf{Ind}_{\mathcal{C}}(S, \mathbb{D})$$

be full and faithful (resp. an equivalence), where both $y_{\mathcal{C}}(X)$ and S are interpreted as discrete $\mathcal{C}$-indexed categories.

Stacks for relative topos theory

The role of stacks in our approach to relative topos theory is two-fold:

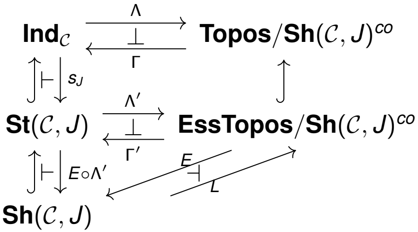
- On the one hand, the notion of stack represents a higher-order categorical generalization of the notion of sheaf. Accordingly, categories of stacks on a site represent higher-categorical analogues of Grothendieck toposes. One can thus expect to be able to lift a number of notions and constructions pertaining to sheaves (resp. Grothendieck toposes) to stacks (resp. categories of stacks on a site).
- On the other hand, stacks on a site $(\mathcal{C}, J)$ generalize internal categories in the topos $\mathbf{Sh}(\mathcal{C}, J)$. Since (ordinary) categories can be endowed with Grothendieck topologies, so stacks on a site can also be endowed with suitable analogues of Grothendieck topologies. This leads to the notion of *site relative to a base topos*, which is crucial for developing relative topos theory.

Remark

Every stack is equivalent to a split stack, and hence to an internal category, but most stacks naturally arising in the mathematical practice are not split (think, for instance, of the canonical stack of a topos).

The big picture

Our theory is based on a network of 2-adjunctions (for any small site $(\mathcal{C}, J)$):



In this diagram s_J denotes the stackification functor, **Topos** the category of Grothendieck toposes and geometric morphisms and **EssTopos** the full subcategory on the essential geometric morphisms.

- The functor E sends an essential geometric morphism $f : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)$ to the object $f_!(1_{\mathscr{E}})$ (where $f_!$ is the left adjoint to the inverse image f^* of f).
- The functor L sends an object P of $\mathbf{Sh}(\mathcal{C}, J)$ to the canonical local homeomorphism $\mathbf{Sh}(\mathcal{C}, J)/P \to \mathbf{Sh}(\mathcal{C}, J)$.

Pseudo-Kan extensions

Proposition

Denote by $\mathbf{Ind}^s_{\mathcal{C}}$ *the sub-2-category of* $\mathbf{Ind}_{\mathcal{C}}$ *of pseudofunctors with values in* **Cat** *(i.e. 'small' $\mathcal{C}$-indexed categories). Consider any functor* $F : \mathcal{C} \to \mathcal{D}$ *and the direct image 2-functor*

$$F^* : \mathbf{Ind}^s_{\mathcal{D}} \to \mathbf{Ind}^s_{\mathcal{C}}$$

which acts by precomposition with F^{op}. *The 2-functor* F^* *has both a left and a right 2-adjoint, denoted respectively by* $\mathrm{Lan}_{F^{\mathrm{op}}}$ *and* $\mathrm{Ran}_{F^{\mathrm{op}}}$, *which act as follows:*

- *for any* D *in* $\mathcal{D}$ *denote by* $\pi^D_F : (D{\downarrow}F) \to \mathcal{C}$ *the canonical projection functor: then for* $\mathbb{E} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, *its image* $\mathrm{Lan}_{F^{\mathrm{op}}}(\mathbb{E}) : \mathcal{D}^{\mathrm{op}} \to \mathbf{Cat}$ *is defined componentwise as*

$$\mathrm{Lan}_{F^{\mathrm{op}}}(\mathbb{E})(D) := \mathrm{colim}_{ps}\left((D{\downarrow}F)^{\mathrm{op}} \xrightarrow{(\pi^D_F)^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathbb{E}} \mathbf{Cat} \right)$$

- *for any* D *in* $\mathcal{D}$ *denote by* $\pi'^D_F : (F{\downarrow}D) \to \mathcal{C}$ *the canonical projection functor: then for* $\mathbb{E} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$, *its image* $\mathrm{Ran}_{F^{\mathrm{op}}}(\mathbb{E}) : \mathcal{D}^{\mathrm{op}} \to \mathbf{Cat}$ *is defined componentwise as*

$$\mathrm{Ran}_{F^{\mathrm{op}}}(\mathbb{E})(D) := \mathrm{lim}_{ps}\left((F{\downarrow}D)^{\mathrm{op}} \xrightarrow{(\pi'^D_F)^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathbb{E}} \mathbf{Cat} \right)$$

Direct and inverse images of stacks

Proposition (O.C. and R.Z.)

Consider two sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ and a functor $F : \mathcal{C} \to \mathcal{D}$.

- *Then F is (J, K)-continuous functor if and only if $F^* : \mathbf{Ind}_{\mathcal{D}} \to \mathbf{Ind}_{\mathcal{C}}$ restricts to a 2-functor $\mathbf{St}(\mathcal{D}, K) \to \mathbf{St}(\mathcal{C}, J)$.*
- *If F is a morphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$, or more generally a (J, K)-continuous functor, it induces a 2-adjunction*

$$\mathbf{St}^s(\mathcal{C}, J) \underset{\mathbf{St}(F)_*}{\overset{\mathbf{St}(F)^*}{\rightleftarrows}} \mathbf{St}^s(\mathcal{D}, K) \quad (\bot),$$

whose pair we shall refer to simply by $\mathbf{St}(F)$.
- *The 2-functor $\mathbf{St}(F)_*$ is called the direct image of stacks along F and acts as the precomposition*

$$F^* := (- \circ F^{\mathrm{op}}) : \mathbf{Ind}_{\mathcal{D}} \to \mathbf{Ind}_{\mathcal{C}};$$

In terms of fibrations, a stack $q : \mathcal{E} \to \mathcal{D}$ is mapped by $\mathbf{St}(F)_*$ to its strict pseudopullback $p : \mathcal{P} \to \mathcal{C}$ along F.

Direct and inverse images of stacks

- The left adjoint $\mathbf{St}(F)^*$ is the inverse image of stacks along F and acts as the composite

$$\mathbf{St}^s(\mathcal{C}, J) \xrightarrow{i_J} \mathbf{Ind}^s_{\mathcal{C}} \xrightarrow{\mathrm{Lan}_{F^{\mathrm{op}}}} \mathbf{Ind}^s_{\mathcal{D}} \xrightarrow{s_K} \mathbf{St}^s(\mathcal{D}, K),$$

where s_K denotes the stackification functor. In terms of fibrations, a stack $p : \mathcal{P} \to \mathcal{C}$ is mapped by $\mathbf{St}(F)^*$ to the stackification of its inverse image $\mathrm{Lan}_{F^{\mathrm{op}}}([p])$ along F, which can be computed as a localization as follows. Consider the fibration of generalized elements

$$(1_{\mathcal{D}} \downarrow (F \circ p)) \xrightarrow{r} \mathcal{D}$$

of the functor $F \circ p$, whose objects are arrows $[d : D \to (F \circ p)(U)]$ of $\mathcal{D}$, and whose morphisms

$$(e, \alpha) : [d' : D' \to (F \circ p)(V)] \to [d : D \to (F \circ p)(U)]$$

are indexed by an arrow $e : D' \to D$ in $\mathcal{D}$ and an arrow $\alpha : V \to U$ in $\mathcal{P}$ such that $(F \circ p)(\alpha) \circ d' = d \circ e$. Consider the class of arrows

$$S := \{(e, \alpha) : [d'] \to [d] \mid (e, \alpha) \ r\text{-vertical}, \ \alpha \text{ cartesian in } \mathcal{P}\} :$$

then

$$\mathrm{Lan}_{F^{\mathrm{op}}}([p]) \simeq (1_{\mathcal{D}} \downarrow (F \circ p))[S^{-1}] \, .$$

Direct and inverse images of stacks

In a similar way to morphisms of sites, comorphisms of sites also induce an adjunction between categories of stacks:

Proposition (O.C. and R.Z.)

Consider a comorphism of sites $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$: it induces a 2-adjunction

$$\mathbf{St}^s(\mathcal{D}, K) \underset{(C_F^{\mathbf{St}})_*}{\overset{(C_F^{\mathbf{St}})^*}{\rightleftarrows}} \mathbf{St}^s(\mathcal{C}, J) \ , \quad \bot$$

whose pair we shall refer to by $C_F^{\mathbf{St}}$.

- *The right adjoint $(C_F^{\mathbf{St}})_*$ acts by restriction of the right pseudo-Kan extension $\mathrm{Ran}_{F^{\mathrm{op}}}$ to stacks;*
- *The left adjoint $(C_F^{\mathbf{St}})^*$ acts as the composite 2-functor*

$$\mathbf{St}^s(\mathcal{D}, K) \xrightarrow{i_K} \mathbf{Ind}^s_{\mathcal{D}} \xrightarrow{F^*} \mathbf{Ind}^s_{\mathcal{C}} \xrightarrow{s_J} \mathbf{St}^s(\mathcal{C}, J),$$

where $F^* := (- \circ F^{\mathrm{op}})$.

- *If F is also continuous the $C_F^{\mathbf{St}}$ also has a left adjoint $(C_F^{\mathbf{St}})_!$ given by the composite 2-functor*

$$\mathbf{St}^s(\mathcal{C}, J) \xrightarrow{i_J} \mathbf{Ind}^s_{\mathcal{C}} \xrightarrow{\mathrm{Lan}_{F^{\mathrm{op}}}} \mathbf{Ind}^s_{\mathcal{D}} \xrightarrow{s_K} \mathbf{St}^s(\mathcal{D}, K) \ .$$

Relative 'presheaf toposes'

Given a $\mathcal{C}$-indexed category $\mathbb{D}$, we denote by $\mathcal{G}(\mathbb{D})$ the fibration on $\mathcal{C}$ associated with it (through the Grothendieck construction) and by $p_{\mathbb{D}}$ the canonical projection functor $\mathcal{G}(\mathbb{D}) \to \mathcal{C}$.

Proposition (O.C. and R.Z.)

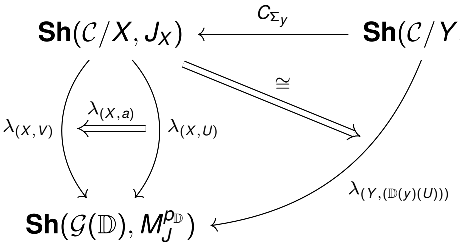
Let $(\mathcal{C}, J)$ be a small-generated site, $\mathbb{D}$ a $\mathcal{C}$-indexed category and $\mathbb{D}^V$ be the opposite indexed category of $\mathbb{D}$ (defined by setting, for each $c \in \mathcal{C}$, $\mathbb{D}^V(c) = \mathbb{D}(c)^{\text{op}}$). Then we have a natural equivalence

$$\mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}}) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathbb{D}^V, \mathcal{S}_{(\mathcal{C},J)}) .$$

This proposition shows that, if $\mathbb{D}$ is a stack, the classifying topos $\mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}})$ of $\mathbb{D}$, which we call the Giraud topos of $\mathbb{D}$, can indeed be seen as the "topos of relative presheaves on $\mathbb{D}$".

Giraud toposes as weighted colimits

We have shown that, for any $\mathbb{D}$, the Giraud topos $C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}}) \to \mathbf{Sh}(\mathcal{C}, J)$ can be naturally seen as a weighted colimit of a diagram of étale toposes over $\mathbf{Sh}(\mathcal{C}, J)$:



where $y : Y \to X$ and $a : U \to V$ are arrows respectively in $\mathcal{C}$ and in $\mathbb{D}(X)$, the legs $\lambda_{(X,U)} : \mathbf{Sh}(\mathcal{C}/X, J_X) \to \mathbf{Sh}(\mathcal{G}(\mathbb{D}), M_J^{p_{\mathbb{D}}})$ of the cocone are the morphisms $C_{\xi_{(X,U)}}$ induced by the morphisms of fibrations $\xi_{(X,U)} : \mathcal{C}/X \to \mathcal{G}(\mathbb{D})$ over $\mathcal{C}$ given by the fibered Yoneda lemma, and the functor $\Sigma_y : \mathcal{C}/Y \to \mathcal{C}/X$ are given by composition with y.

The fundamental adjunction

The universal property of the above weighted colimit yields a fundamental 2-adjunction between cloven fibrations over $\mathcal{C}$ and toposes over $\mathbf{Sh}(\mathcal{C}, J)$:

Theorem (O.C and R.Z.)

For any small-generated site $(\mathcal{C}, J)$, the two pseudofunctors

$$\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{cFib}_{\mathcal{C}} \xrightarrow{\mathfrak{G}} \mathbf{Com}/(\mathcal{C}, J) \xrightarrow{C_{(-)}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J),$$

$$\Big[[p : \mathcal{D} \to \mathcal{C}] \xrightarrow{(F,\phi)} [q : \mathcal{E} \to \mathcal{C}]\Big] \mapsto \Big[[\mathrm{Gir}_J(p)] \xrightarrow{(C_F, C_\phi)} [\mathrm{Gir}_J(q)]\Big],$$

and

$$\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)} : \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Ind}_{\mathcal{C}} \simeq \mathbf{cFib}_{\mathcal{C}},$$

$$[E : \mathscr{E} \to \mathbf{Sh}(\mathcal{C}, J)] \mapsto \big[\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J)(\mathbf{Sh}(\mathcal{C}/-, J_{(-)}), [E]) : \mathcal{C}^{\mathrm{op}} \to \mathbf{CAT}\big]$$

are the two components of a 2-adjunction

$$\mathbf{cFib}_{\mathcal{C}} \underset{\Gamma_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}}{\overset{\Lambda_{\mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C},J)}}{\rightleftarrows}} \mathbf{Topos}^{co}/\mathbf{Sh}(\mathcal{C}, J) \quad (\bot)$$

Remark

Since $\mathrm{Gir}_J(p) \simeq \mathbf{Ind}_{\mathcal{C}}(\mathcal{D}^V, \mathcal{S}_{(\mathcal{C},J)})$, the canonical stack $\mathcal{S}_{(\mathcal{C},J)}$ has a similar behavior to that of a dualizing object for the adjunction $\Lambda \dashv \Gamma$.

The discrete setting

The restriction of our fundamental adjunction in the setting of presheaves (that is, discrete fibrations) will yield a generalization to the context of arbitrary sites of the classical adjunction

$$\mathbf{Psh}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathbf{Top}/X \quad \text{(with } \Lambda \dashv \Gamma\text{)} .$$

between presheaves on a topological space X and bundles over it.

This adjunction can be notably applied to the theory of sheaves, leading to fibrational descriptions of the sheafification functor, as well as of direct and inverse images of sheaves.

Relative sheaf toposes

As any Grothendieck topos is a subtopos of a presheaf topos, so any relative topos should be a subtopos of a relative presheaf topos. This motivates the following

Definition

Let $(\mathcal{C}, J)$ be a small-generated site. A site relative to $(\mathcal{C}, J)$ is a pair consisting of a $\mathcal{C}$-indexed category $\mathbb{D}$ and a Grothendieck topology K on $\mathcal{G}(\mathbb{D})$ which contains the Giraud topology $M_J^{p_{\mathbb{D}}}$.

The topos of sheaves on such a relative site $(\mathbb{D}, K)$ is defined to be the geometric morphism

$$C_{p_{\mathbb{D}}} : \mathbf{Sh}(\mathcal{G}(\mathbb{D}), K) \to \mathbf{Sh}(\mathcal{C}, J)$$

induced by the comorphism of sites $p_{\mathbb{D}} : (\mathcal{G}(\mathbb{D}), K) \to (\mathcal{C}, J)$.

Remark

Not every Grothendieck topology on K can be generated starting by horizontal or vertical data (that is, by sieves generated by cartesian arrows or entirely lying in some fiber), but many important relative topologies naturally arising in practice are of this form.

Examples of relative topologies

- The relative topology on the canonical stack of a geometric morphism (which allows one to represent *any* relative topos as the topos of sheaves on a relative site).
- The Giraud topology is an example of a relative topology generated by horizontal data.
- The total topology of a fibered site, in the sense of Grothendieck, is generated by vertical data.

We have shown that, for a wide class of relative topologies generated by horizontal and vertical data, one can describe bases for them consisting of multicompositions of horizontal families with vertical families.

A question of Grothendieck

As recently brought to the public attention by Colin McLarty, Grothendieck expressed, in his 1973 Buffalo lectures, the aspiration of viewing any object of a topos geometrically as an étale space over the terminal object:

> *The intuition is the following: viewing objects of a topos as being something like étale spaces over the final object of the topos, and the induced topos over an object as just the object itself. That is I think the way one should handle the situation.*
>
> *It's a funny situation because in strict terms, you see, the language which I want to push through doesn't make sense. But of course there are a number of mathematical statements which substantiate it.*

Given his conception of *gros* and *petit* toposes, we can more broadly interpret his wish as that for a framework allowing one to think geometrically about any topos, that is, as it were a 'petit' topos related to a 'gros' topos by a local retraction.

Local morphisms

Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be local if f_* has a fully faithful right adjoint.

Theorem (O.C.)

Let $F : \mathcal{D} \to \mathcal{C}$ be a bimorphism of sites $(\mathcal{D}, K) \to (\mathcal{C}, J)$. Then:

(i) *The geometric morphism $C_F : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$ is essential, and*

$$(C_F)_! \cong \mathbf{Sh}(F)^* \dashv \mathbf{Sh}(F)_* \cong (C_F)^* = D_F := (- \circ F^{\mathrm{op}}) \dashv (C_F)_*$$

(ii) *The morphism $\mathbf{Sh}(F) : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ is local if and only if C_F is an inclusion, that is, if and only if F is K-faithful and K-full. In this case, the morphisms C_F and $\mathbf{Sh}(F)$ realize the topos $\mathbf{Sh}(\mathcal{D}, K)$ as a (coadjoint) retract of $\mathbf{Sh}(\mathcal{C}, J)$ in* **Topos**.

Gros and *petit* toposes

Pairs of *gros* and *petit* toposes are important for several reasons. Morally, a *petit* topos is thought of as a generalized space, while a *gros* topos is conceived as a category of spaces.

In fact, one advantage of *gros* toposes is that they are associated with sites which tend to have better categorical properties than those of the site presenting the *petit* topos.

Still, *gros* and *petit* toposes in a given pair are homotopically equivalent (as they are related by a local retraction), whence they share the same cohomological invariants.

The above result can be notably applied to construct pairs of *gros* and *petit* toposes starting from a (K-)full and (K-)faithful bimorphism of sites

$$(\mathcal{D}, K) \to (\mathcal{T}/T_{\mathcal{D}}, E_{T_{\mathcal{D}}}),$$

where $\mathcal{T}$ is a category endowed with a Grothendieck topology E, $T_{\mathcal{D}}$ is an object of $\mathcal{T}$ and $E_{T_{\mathcal{D}}}$ is the Grothendieck topology induced on $(\mathcal{T}/T_{\mathcal{D}})$ by E.

Every Grothendieck topos is a 'small topos'

We define a Grothendieck topology $J^{\text{ét}}$ on **Topos**, which we call the étale cover topology, by postulating that a sieve on a topos $\mathcal{E}$ is $J^{\text{ét}}$-covering if and only if it contains a family $\{\mathcal{E}/A_i \to \mathcal{E} \mid i \in I\}$ of canonical local homeomorphisms such that the family of arrows $\{!_{A_i} : A_i \to 1_{\mathcal{E}} \mid i \in I\}$ is epimorphic in $\mathcal{E}$.

The functor L is a J-full and J-faithful bimorphism of sites

$$(\mathcal{C}, J) \to (\mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J), J^{\text{ét}}_{\mathbf{Sh}(\mathcal{C},J)}) \,.$$

So, by the above result, the 'petit' topos $\mathbf{Sh}(\mathcal{C}, J)$ identifies with a coadjoint retract of the 'big' topos $\mathbf{Sh}(\mathbf{Topos}/\mathbf{Sh}(\mathcal{C}, J), J^{\text{ét}}_{\mathbf{Sh}(\mathcal{C},J)}) \simeq \mathbf{Sh}(\mathbf{Topos}, J^{\text{ét}})/l(\mathbf{Sh}(\mathcal{C}, J))$ (in a suitable Grothendieck universe) via the local morphism $\mathbf{Sh}(L)$ and the essential inclusion C_L.

This shows that *every* Grothendieck topos can be naturally regarded as a 'petit' topos embedded in an associated 'gros' topos, and that this embedding allows one to view any object of the original topos as an étale morphism to the terminal object in the associated 'gros' topos, thus providing a solution to Grothendieck's problem.

Future developments

Our notion of relative site will play a key role in our future development the theory of *relative toposes*.

We expect the development of this theory to parallel that of the classical theory; indeed, by using a general stack semantics, we plan to introduce, in a canonical, not *ad hoc* way, natural generalizations to the relative setting of the classical notions of morphism and comorphism of sites, flat functors, separating sets for a topos, denseness conditions etc.

This will notably lead us to relative versions, in the language of stacks (or, more generally, of indexed categories), of Giraud's and Diaconescu's theorems, as well as to a theory of classifying toposes of (higher-order) relative geometric theories.

Towards relative geometric logic

Indeed, the geometric approach to relative toposes which we have developed so far has a logical counterpart, which we may call relative geometric logic.

In its classical formulation, geometric logic does not have parameters embedded in its formalism; still, it is possible to introduce them without changing its degree of expressivity.

In a relative setting, parameters are fundamental if one wants to reason geometrically and use fibrational techniques. In fact, the semantics of stacks involves parameters in an essential way.

It turns out that the logical framework corresponding to relative toposes is a kind of fibrational, higher-order parametric logic in which it is possible to express a great number of higher-order constructions by using the parameters belonging to the base topos.

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