Extending the topological presheaf-bundle adjunction to sites and toposes

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joint work with Olivia Caramello

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Content of the talk

Known result

Consider a topological space X: there is an adjunction

$$\mathrm{Psh}(X) \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathrm{Top}/X \quad (\Lambda \dashv \Gamma)$$

Generalization

Consider a site $(\mathcal{C}, J)$: there is an adjunction

$$[\mathcal{C}^{op}, \mathrm{Set}] \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \mathrm{Topos}^s/\mathrm{Sh}(\mathcal{C}, J) \quad (\Lambda \dashv \Gamma)$$

References

The content of this talk is an extract of

- Caramello, O. and Zanfa, R., *Relative topos theory via stacks*, 2021

The topological presheaf-bundle adjunction can be found for instance in Sections II.4, II.5 and II.6 of *Sheaves in Geometry and Logic: An Introduction to Topos Theory*, by S. Mac Lane and I. Moerdijk; the localic adjunction is implicit in Exercises from 9 to 12 in Chapter IX of the same book.

Preliminaries: the topological adjunction (1)

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Local sections functor: Γ

It maps a bundle $p : E \to X$ to its presheaf of local sections Γ_p, defined on any open $U \subseteq X$ as

$$\Gamma_p(U) := \{f : U \to E \mid f \text{ continuous, } p \circ f = i_U : U \hookrightarrow X\}$$

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- the composite $\Gamma \circ \Lambda$ corresponds to the sheafification functor

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Conclusions:

- (pre)sheaves over X can be thought as a category of spaces
- geometric understanding of the sheafification functor

Preliminaries: the localic adjunction

The topological adjunction is essentially point-free, and extends to locales:

Presheaf-bundle adjunction for locales:

$$\begin{array}{ccc} \mathsf{Psh}(L) & \underset{\Gamma}{\overset{\Lambda}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & \mathsf{Locale}/L \\ \uparrow & & \uparrow \\ \mathsf{Sh}(L) & \overset{\sim}{\longrightarrow} & \mathsf{Etale}/L \end{array}$$

where in particular the composite $\Gamma \circ \Lambda$ coincides with the sheafification

$$\mathrm{a} : \mathsf{Psh}(L) \to \mathsf{Sh}(L)$$

Toposes generalize spaces (1)

We replace topological spaces/locales with sites:

$$\begin{array}{rrcc} X, & L & \rightsquigarrow & (\mathcal{C}, J) \\ \mathrm{Top}/X, & \mathrm{Locale}/L & \rightsquigarrow & \end{array}$$

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Geometric morphisms generalize continuous maps.
To define local sections we need to take care of a size issue:

Relatively small toposes

$\mathscr{E} \to \mathsf{Sh}(\mathcal{C}, J)$ is *small relatively to* $\mathsf{Sh}(\mathcal{C}, J)$ if for each X in $\mathcal{C}$ the category of geometric morphisms

$$\mathsf{Topos}/\mathsf{Sh}(\mathcal{C}, J)(\mathsf{Sh}(\mathcal{C}, J)/\ell_J(X), \mathscr{E})$$

is a set, up to equivalence of geometric morphisms (ℓ_J being the canonical functor $\mathcal{C} \to \mathsf{Sh}(\mathcal{C}, J)$).
We denote by $\mathsf{Topos}^s/\mathsf{Sh}(\mathcal{C}, J)$ the full subcategory of $\mathsf{Topos}/\mathsf{Sh}(\mathcal{C}, J)$ whose objects are the relatively small toposes.

Toposes generalize spaces (2)

Étale toposes

A topos $\mathscr{E} \to \mathrm{Sh}(\mathcal{C}, J)$ is *étale* (or a *local homeomorphism*) if it is equivalent to one of the form

$$\mathrm{Sh}(\mathcal{C}, J)/F \to \mathrm{Sh}(\mathcal{C}, J).$$

Denote the category of étale toposes over $\mathrm{Sh}(\mathcal{C}, J)$ by $\mathrm{Etale}/\mathrm{Sh}(\mathcal{C}, J)$.

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We can now complete the picture:

$$\begin{array}{rrcc} X, & L & \rightsquigarrow & (\mathcal{C}, J) \\ \mathrm{Top}/X, & \mathrm{Locale}/L & \rightsquigarrow & \mathrm{Topos}^s/\mathrm{Sh}(\mathcal{C}, J) \\ \mathrm{Etale}/X, & \mathrm{Etale}/L & \rightsquigarrow & \mathrm{Etale}/\mathrm{Sh}(\mathcal{C}, J) \end{array}$$

Replacing bundle of germs and local sections

Local sections functor $\Gamma : \mathsf{Topos}^s/\mathsf{Sh}(\mathcal{C}, J) \to [\mathcal{C}^{op}, \mathsf{Set}]$

It maps $\mathscr{E} \to \mathsf{Sh}(\mathcal{C}, J)$ to the presheaf

$$\mathsf{Topos}/\mathsf{Sh}(\mathcal{C}, J)(\mathsf{Sh}(\mathcal{C}, J)/\ell_J(-), \mathscr{E}) : \mathcal{C}^{op} \to \mathsf{Set}$$

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'Bundle of germs' functor $\Lambda : [\mathcal{C}^{op}, \mathrm{Set}] \to \mathrm{Topos}^s/\mathrm{Sh}(\mathcal{C}, J)$

It maps a presheaf $P : \mathcal{C}^{op} \to \mathrm{Set}$ to the geometric morphism

$$C_{\pi_P} : \mathrm{Sh}(\textstyle\int P, J_P) \to \mathrm{Sh}(\mathcal{C}, J)$$

where $\pi_P : \int P \to \mathcal{C}$ is the fibration associated to P and J_P is **Giraud's topology** on $\int P$.

The presheaf-bundle adjunction for sites (1)

Lemma on colimits of toposes:

$$\mathsf{Sh}(\textstyle\int P, J_P) \simeq \operatorname{colim}\left(\textstyle\int P \xrightarrow{\pi_P} \mathcal{C} \xrightarrow{\mathsf{Sh}(\mathcal{C},J)/\ell_J(-)} \mathsf{Topos}/\mathsf{Sh}(\mathcal{C}, J)\right)$$

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Then for $P : \mathcal{C}^{op} \to \mathrm{Set}$ and $\mathscr{E} \to \mathrm{Sh}(\mathcal{C}, J)$ we have that

$$\Lambda(P) := \mathrm{Sh}(\textstyle\int P, J_P) \to \mathscr{E}$$

over $\mathrm{Sh}(\mathcal{C}, J)$

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are indexed by elements of $\int P$, i.e. a natural transformation

$$P \to \mathbf{Topos}/\mathsf{Sh}(\mathcal{C}, J)(\mathsf{Sh}(\mathcal{C}, J)/\ell_J(-), \mathscr{E}) := \Gamma(\mathscr{E})$$

This shows that $\Lambda \dashv \Gamma$.

The presheaf-bundle adjunction for sites (2)

Lemmas on étale toposes:

$$\mathsf{Topos}/\mathscr{E}(\mathscr{E}/A, \mathscr{E}/B) \simeq \mathscr{E}(A, B)$$

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Fixed points in $[\mathcal{C}^{op}, \mathsf{Set}]$ **are precisely the** J**-sheaves, and** $\Gamma\Lambda \simeq \mathrm{a}_J$

The presheaf-bundle adjunction for sites (3)

Presheaf-bundle adjunction for sites:

$$\begin{array}{ccc}
[\mathcal{C}^{op}, \mathsf{Set}] & \underset{\Gamma}{\overset{\Lambda}{\rightleftarrows}} \; \bot & \mathsf{Topos}^{s}/\mathsf{Sh}(\mathcal{C}, J) \\
\uparrow & & \uparrow \\
\mathsf{Sh}(\mathcal{C}, J) & \overset{\sim}{\longrightarrow} & \mathsf{Etale}/\mathsf{Sh}(\mathcal{C}, J)
\end{array}$$

The composite $\Gamma \circ \Lambda$ is equivalent to the sheafification functor

$$\mathrm{a}_J : [\mathcal{C}^{op}, \mathsf{Set}] \to \mathsf{Sh}(\mathcal{C}, J)$$

Further aspects of the presheaf-bundle adjunction

- 'going up': the 2-categorical adjunction between $\mathcal{C}$-indexed categories and toposes over $\mathsf{Sh}(\mathcal{C}, J)$
- 'going down': descriptions of the sheafification in terms of sites and morphisms/comorphisms, and a geometric perspective on sheafification
- 'crossing the bridge': preorder sites, an example in which the topos-theoretic datum vanishes

2-categorical adjunction

The presheaf-bundle adjunction for sites is the 1-dimensional truncation of the 2-categorical adjunction

$$[\mathcal{C}^{op}, \mathrm{Cat}]_{ps} \underset{\Gamma}{\overset{\Lambda}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Topos}^{co}/\mathrm{Sh}(\mathcal{C}, J),$$

where Γ acts again as a local sections functor, while Λ maps a $\mathcal{C}$-indexed category $\mathbb{D} : \mathcal{C}^{op} \to \mathrm{Cat}$ to its *Giraud topos*

$$\mathrm{Gir}_J(\mathbb{D}) := \mathrm{Sh}(\mathcal{G}(\mathbb{D}), J_{\mathbb{D}})$$

Sheafification in terms of sites

$\mathrm{a}_J \cong \Gamma \circ \Lambda$ is a geometric interpretation of a_J as local sections:

$$\mathrm{a}_J(P)(X) := \mathsf{Topos}/\mathsf{Sh}(\mathcal{C}, J)(\mathsf{Sh}(\mathcal{C}, J)/\ell_J(X), \mathsf{Sh}(\textstyle\int P, J_P))$$

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$$(\mathcal{C}/X, J_X) \to ([(\textstyle\int P)^{op}, \mathsf{Set}], \widehat{J_P})$$

- As given by a J-covering family $\{Y_i \to X\}$ and a family of morphisms of fibrations

$$\mathcal{C}/Y_i \to \textstyle\int P$$

Presheaf-bundle adjunction for preorders (1)

Lemmas on locales:

$$\mathsf{Locale}/L \hookrightarrow \mathsf{Topos}^s/\mathsf{Sh}(L)$$

For a preorder site $(\mathcal{C}, J)$ and a presheaf $P : \mathcal{C}^{op} \to \mathsf{Set}$,

$$\mathsf{Sh}(\mathcal{C}, J) \simeq \mathsf{Sh}(Id_J(\mathcal{C})), \ \mathsf{Sh}(\textstyle\int P, J_P) \simeq \mathsf{Sh}(Id_{J_P}(\textstyle\int P))$$

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Consider P a presheaf over a locale L: then

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$$\begin{aligned} \mathrm{a}_J(P)(X) &\simeq \mathrm{Topos}/\mathrm{Sh}(L)(\mathrm{Sh}(L)/\ell_J(X), \mathrm{Sh}(\textstyle\int P, J_P)) \\ &\simeq \mathrm{Locale}/L(Sub(\langle X\rangle_J), Id_{J_P}(\textstyle\int P)), \end{aligned}$$

$Sub(\langle X\rangle_J)$ being the locale of subideals of the J-ideal generated by X.

Presheaf-bundle adjunction for preorders (2)

Presheaf-bundle adjunction for preorders

$$[\mathcal{C}^{op}, \mathrm{Set}] \underset{\Gamma}{\overset{\Lambda}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathrm{Locale}/Id_J(\mathcal{C})$$

inducing equivalences

$$[\mathcal{C}^{op}, \mathrm{Set}] \simeq \mathrm{Etale}(\mathcal{C}),\ \ \mathrm{Sh}(\mathcal{C}, J) \simeq \mathrm{Etale}/Id_J(\mathcal{C}) \simeq \mathrm{Etale}_J(\mathcal{C})$$

The latter equivalence, formulated for a suitable notion of J-étale map over the preorder site $(\mathcal{C}, J)$, generalizes a result by J. Hemelaer in *A Topological Groupoid Representing the Topos of Presheaves on a Monoid*, 2020.