The Logic and Geometry of Localic Morphisms

Joshua Wrigley
Università degli Studi dell'Insubria
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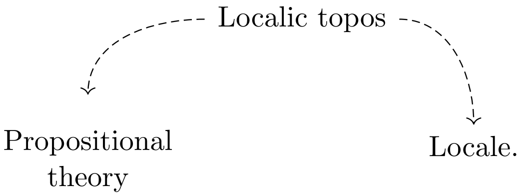
Propositional intuition



Propositional intuition

Localic topos

- Propositional theory
- Locale.

Localic morphism

- Geometric theory
- Internal locale.

Localic morphisms

Definition

A geometric morphism $\mathcal{F} \to \mathcal{E}$ is *localic* if for every $F \in \mathcal{F}$, there exist $F' \in \mathcal{F}, E \in \mathcal{E}$ such that:

$$F \rightarrowtail F' \twoheadleftarrow f^*(E).$$

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Examples

(1) Every inclusion is localic.

(2) $\mathbf{Loc} \xrightarrow{\mathbf{Sh}} \mathfrak{BTop}/\mathbf{Sets}$ is full and faithful, reflective, and $\mathbf{Sh}(f)$ is localic for each morphism $X \xrightarrow{f} Y \in \mathbf{Loc}$.

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(3) Similarly, $\mathbf{Loc}(\mathcal{E}) \xrightarrow{\mathbf{Sh}_{\mathcal{E}}} \mathfrak{BTop}/\mathcal{E}$ is full and faithful, reflective, and $\mathbf{Sh}(f)$ is localic for each morphism $X \xrightarrow{f} Y \in \mathbf{Loc}(\mathcal{E})$.

(4) A morphism $f\colon \mathcal{F} \to \mathcal{E}$ is localic if and only if $\mathcal{F} \simeq \mathbf{Sh}_{\mathcal{E}}(f_*(\Omega_{\mathcal{F}}))$.

Internal locales

Definition An *internal locale* of $[\mathcal{C}^{op}, \mathbf{Sets}]$, where $\mathcal{C}$ has finite limits, is a functor $F \colon \mathcal{C} \to \mathbf{Loc}$ such that each $F(f)$ has a left adjoint $\exists_f$ satisfying:

$$\exists_f(U \wedge F(f)^{-1}(V)) = \exists_f(U) \wedge V,$$

And for each pullback in $\mathcal{C}$:

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle h} \\ c & \xrightarrow{k} & d, \end{array}$$

The diagram commutes:

$$\begin{array}{ccc} F(a) & \xleftarrow{F(f)^{-1}} & F(b) \\ \big\downarrow{\scriptstyle \exists_g} & & \big\downarrow{\scriptstyle \exists_h} \\ F(c) & \xleftarrow{F(k)^{-1}} & F(d). \end{array}$$

Localic expansions of theories

Definition A *localic expansion* of $\mathbb{T}$ is a theory $\mathbb{T}' \supseteq \mathbb{T}$ in a signature $\Sigma' \subseteq \Sigma$ that adds new relation and function symbols (*but no new sorts!*).

Theorem *If $\mathbb{T}'$ is a localic expansion of $\mathbb{T}$, there is a localic morphism $p\colon \mathcal{E}_{\mathbb{T}'} \to \mathcal{E}_{\mathbb{T}}$.*

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Definition The *category of relabellings of sorts* Sort_Σ has:

(1) As objects finite strings of variables $\vec{x}$ in Σ,

(2) As arrows $\vec{y} \xrightarrow{\sigma} \vec{x}$ maps that respect sorts.

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As $\mathcal{E}_{\mathbb{O}_\Sigma} \simeq [\mathrm{Sort}_\Sigma, \mathbf{Sets}]$, $\mathcal{E}_{\mathbb{T}}$ is the topos of sheaves on an internal locale of $[\mathrm{Sort}_\Sigma, \mathbf{Sets}]$.

Localic expansions of theories

Toposes localic over $[\mathrm{Sort}_\Sigma, \mathbf{Sets}]$

Geometric theory in signature Σ

Internal locale of $[\mathrm{Sort}_\Sigma, \mathbf{Sets}]$.

Syntactic sites

Let $\mathbb{T}$ be a geometric theory.

Definition

The *syntactic category* $\mathcal{C}_{\mathbb{T}}$ has:

(1) As objects $\{\vec{x} : \varphi\}$,

(2) As arrows $\{\vec{x} : \varphi\} \xrightarrow{[\theta]} \{\vec{y} : \psi\}$ provable equivalence classes of formulae $[\theta]$ such that:

$$\varphi \vdash_{\vec{x}} \exists \vec{y}\, \theta,\ \ \theta \vdash_{\vec{x},\vec{y}} \varphi \wedge \psi,\ \ \theta \wedge \theta[\vec{z}/\vec{y}] \vdash_{\vec{x},\vec{y},\vec{z}} \vec{y} = \vec{z}.$$

Definition

The *syntactic topology* $J_{\mathbb{T}}$ has S, a sieve on $\{\vec{y} : \psi\}$, covering if and only if:

$$\psi \vdash_{\vec{y}} \bigvee_{\theta \in S} \exists \vec{x}\, \theta.$$

Theorem

$\mathcal{E}_{\mathbb{T}} \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$.

Proposition

The topology $J_{\mathbb{T}}$ is subcanonical.

Substitutive syntactic site

Definition

To $\mathbb{T}$, we associate $F^{\mathbb{T}} \colon \mathrm{Sort}_{\Sigma}^{op} \to \mathbf{Loc}$ where:

(1) $F^{\mathbb{T}}(\vec{x})$ is the formalae $\{\, \vec{x} : \varphi \,\}$ ordered by $\varphi \vdash_{\vec{x}} \psi$,

(2) $F^{\mathbb{T}}(\sigma) \colon F^{\mathbb{T}}(\vec{x}) \to F^{\mathbb{T}}(\vec{y})$ is the frame homomorphism:

$$\sigma^{-1}(\{\, \vec{y} : \psi \,\}) = \{\, \vec{x} : \psi[\vec{x}/_{\sigma}\vec{y}] \,\}.$$

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Remarks

(1) $F^{\mathbb{T}}\colon \mathrm{Sort}_\Sigma^{op} \to \mathbf{Loc}$ forms an internal locale of $[\mathrm{Sort}_\Sigma, \mathbf{Sets}]$: it satisfies the Beck Chevalley and Frobenius conditions.

(2) The left and right adjoints of σ^{-1} are:

$$\exists_\sigma \{\, \vec{x} : \varphi \,\} = \{\, \vec{y} : \exists \vec{x}\, \varphi \wedge \sigma(\vec{y}) = \vec{x} \,\},$$
$$\forall_\sigma \{\, \vec{x} : \varphi \,\} = \{\, \vec{y} : \forall \vec{x}\, \varphi \wedge \sigma(\vec{y}) = \vec{x} \,\}.$$

Substitutive syntactic site

The category $\mathrm{Sort}_\Sigma \rtimes F^{\mathbb{T}}$ is a *substitutive syntactic site* whose objects are $\{\, \vec{x} : \varphi \,\}$ and arrows are $\sigma \colon \vec{y} \to \vec{x}$ such that $\varphi \vdash_{\vec{x}} \psi[\vec{x}/_\sigma \vec{y}]$.

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In the induced topology on $\mathrm{Sort}_\Sigma \rtimes F^{\mathbb{T}}$, a family of arrows $\{\, \{\, \vec{x}_i : \varphi_i \,\} \xrightarrow{\sigma_i} \{\, \vec{y} : \psi \,\} \,\}$ is covering iff:

$$\psi \vdash_{\vec{y}} \bigvee_i \exists \vec{x}_i \, \varphi_i \wedge \sigma_i(\vec{y}) = \vec{x}_i.$$

Call this $K_{\mathbb{T}}$, then: $\mathbf{Sh}_{\mathcal{E}_{\mathbb{O}_\Sigma}}(F^{\mathbb{T}}) \simeq \mathbf{Sh}(\mathrm{Sort}_\Sigma^{op} \rtimes F^{\mathbb{T}}, K_{\mathbb{T}})$.

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Proposition

The topology $K_{\mathbb{T}}$ is generated by the species:

(a) $\{\, \{\, \vec{x} : \varphi_i \,\} \to \{\, \vec{x} : \bigvee_i \varphi_i \,\} \,\}_{i \in I}$,

(b) $\{\, \vec{x} : \varphi \,\} \to \{\, \vec{y} : \exists \vec{x} \, \varphi \wedge \sigma(\vec{y}) = \vec{x} \,\}$.

Subsitutive syntactic site

Theorem *The functor* $\eta\colon (\mathrm{Sort}_\Sigma \rtimes F^{\mathbb{T}}, K_{\mathbb{T}}) \to (\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ *is a dense morphism of sites, where:*

$$\eta(\{\,\vec{x} : \varphi\,\}) = \{\,\vec{x} : \varphi\,\},$$

$$\eta\left(\{\,\vec{x} : \varphi\,\} \xrightarrow{\sigma} \{\,\vec{y} : \psi\,\}\right) = [\varphi \wedge \sigma(\vec{y}) = \vec{x}].$$

Corollary $\mathbf{Sh}(\mathrm{Sort}_\Sigma \rtimes F^{\mathbb{T}}, K_{\mathbb{T}}) \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}}) \simeq \mathcal{E}_{\mathbb{T}}.$

Dense morphism of sites

Definition A *dense morphism of sites* $(\mathcal{C}, J) \to (\mathcal{D}, K)$ is a functor $F: \mathcal{C} \to \mathcal{D}$ such that:

(1) S is a J-covering if and only if $F(S)$ is K-covering.

(2) For every $d \in \mathcal{D}$, there exists a K-covering family of morphisms $F(c_i) \to d$.

(3) For every $c_1, c_2 \in \mathcal{C}$ and arrow $g: F(c_1) \to F(c_2)$ in $\mathcal{D}$, there is a J-covering family $f_i: c_i' \to c_1$ and a family of arrows $k_i: c_i' \to c_2$ such that $g \circ F(f_i) = F(k_i)$.

(4) For any arrows $f_1, f_2: c_1 \to c_2$ in $\mathcal{C}$ such that $F(f_1) = F(f_2)$, there exists a J-covering family of arrows $k_i: c_i' \to c_1$ such that $f_1 \circ k_i = f_2 \circ k_i$.

Theorem *A dense morphism of sites induces an equivalence of categories* $\mathbf{Sh}(\mathcal{C}, J) \simeq \mathbf{Sh}(\mathcal{D}, K)$.

Proof of theorem

(1) Immediate by definition,

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(2) Immediate since η is surjective on objects,

(3) Let $\{\,\vec{x} : \varphi\,\} \xrightarrow{[\theta]} \{\,\vec{y} : \psi\,\} \in \mathcal{C}_{\mathbb{T}}$. Consider:

$$\{\,\vec{x}', \vec{y}' : \theta(\vec{x}', \vec{y}')\,\} \xrightarrow{[\theta(\vec{x}',\vec{y}')\wedge\vec{x}'=\vec{x}]} \{\,\vec{x} : \varphi\,\} \xrightarrow{[\theta(\vec{x},\vec{y})]} \{\,\vec{y} : \psi\,\}.$$

$$\{\,\vec{x}', \vec{y}' : \theta(\vec{x}', \vec{y}')\,\} \xrightarrow{[\exists\vec{x}\,\theta(\vec{x}',\vec{y}')\wedge\vec{x}'=\vec{x}\wedge\theta(\vec{x},\vec{y})]} \{\,\vec{y} : \psi\,\}$$

Follows from:

$$\varphi \vdash_{\vec{x}} \exists\vec{y}\,\theta, \; \exists\vec{x}\,\theta(\vec{x}', \vec{y}')\wedge\vec{x}' = \vec{x}\wedge\theta(\vec{x}, \vec{y}) \equiv \theta(\vec{x}', \vec{y}')\wedge\vec{y}' = \vec{y}.$$

Proof of theorem

Proof

(4) if two relabellings $\sigma, \tau \colon \{\, \vec{x} : \varphi \,\} \to \{\, \vec{y} : \psi \,\}$ are such that $\eta(\sigma) = \eta(\tau)$ then:

$$\varphi \vdash_{\vec{x}} \sigma(y_i) = \tau(y_i) \text{ for each } y_i \in \vec{y}.$$

Consider the commutative diagram:

$$\{\, \vec{x}' : \varphi[\vec{x}'/_{\pi}\vec{x}] \,\} \xrightarrow{\ \pi\ } \{\, \vec{x} : \varphi \,\} \underset{\tau}{\overset{\sigma}{\rightrightarrows}} \{\, \vec{y} : \psi \,\},$$

Where $\pi \colon \vec{x} \to \vec{x}'$ is the coequalizer of $\sigma, \tau \colon \vec{y} \to \vec{x}$. we aim to show that π is a covering arrow, but the required sequent, $\varphi \vdash_{\vec{x}} \exists \vec{x}' \, \varphi[\vec{x}'/_{\pi}\vec{x}] \wedge \pi(\vec{x}) = \vec{x}'$ follows from:

$$\varphi \vdash_{\vec{x}} \sigma(y_i) = \tau(y_i) \text{ for each } y_i \in \vec{y}.$$

Further remarks

Proposition *If* $\mathbb{T}'$ *is a localic expansion of* $\mathbb{T}$*, then there is a morphism of locales* $\alpha\colon F^{\mathbb{T}'} \Rightarrow F^{\mathbb{T}}$ *internal to* $[\mathrm{Sort}_{\Sigma}, \mathbf{Sets}]$ *whence* $\mathcal{E}_{\mathbb{T}'} \to \mathcal{E}_{\mathbb{T}}$ *is localic.*

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Proposition *If $\mathbb{T}'$ is a localic expansion of $\mathbb{T}$, then there is a morphism of locales $\alpha\colon F^{\mathbb{T}'} \Rightarrow F^{\mathbb{T}}$ internal to $[\mathrm{Sort}_\Sigma, \mathbf{Sets}]$ whence $\mathcal{E}_{\mathbb{T}'} \to \mathcal{E}_{\mathbb{T}}$ is localic.*

Definition A *quotient theory* $\mathbb{T}'$ of $\mathbb{T}$ is a theory in the same signature such that $\mathbb{T}' \supseteq \mathbb{T}$.

Proposition *There is a correspondence:*

$$\begin{aligned} \textit{Subtopoi } \mathcal{F} \hookrightarrow \mathcal{E}_{\mathbb{T}} &\iff \textit{Inclusions of internal} \\ &\qquad \textit{sublocales } \mathbb{L} \hookrightarrow F^{\mathbb{T}}, \\ &\iff \textit{Quotient theories } \mathbb{T}' \supseteq \mathbb{T}. \end{aligned}$$

The syntactic site as a completion

Theorem

(1) The syntactic category $\mathcal{C}_{\mathbb{T}}$ *is the full subcategory of* $\ell(\{\,\vec{x} : \varphi\,\}) \in \mathbf{Sh}(\mathrm{Sort}_{\Sigma} \rtimes F^{\mathbb{T}}, K_{\mathbb{T}}) \simeq \mathcal{E}_{\mathbb{T}}$

(2) The syntactic topology $J_{\mathbb{T}}$ *is the restriction of the canonical topology on* $\mathcal{E}_{\mathbb{T}}$*.*

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(2) The syntactic topology $J_{\mathbb{T}}$ is the restriction of the canonical topology on $\mathcal{E}_{\mathbb{T}}$.

Proof

The topology $J_{\mathbb{T}}$ is subcanonical, so $\mathcal{C}_{\mathbb{T}}$ is a full subcategory of $\mathcal{E}_{\mathbb{T}}$ and $J_{\mathbb{T}} = J_{\mathcal{E}}^{can}|_{\mathcal{C}_{\mathbb{T}}}$.

Thank you for listening!

Localic morphisms:

[1] P.T. Johnstone, *Factorization Theorems for Geometric Morphisms, I*, Cahiers top. et gèom. diff. Vol. XXII-1 (1981).

Internal locales:

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