# A topos-theoretic view of difference algebra

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# Ritt-style difference algebra

### Ritt 1930s

Difference algebra is the study of rings and modules with added endomorphisms.

# Difference categories à la Ritt

Let $\mathscr{C}$ be a category. Define its difference category

$$\boldsymbol{\sigma}\text{-}\mathscr{C}$$

- objects are pairs
$$(X, \sigma_X),$$
where $X \in \mathscr{C}$, $\sigma_X \in \mathscr{C}(X, X)$;
- a morphism $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is a commutative diagram in $\mathscr{C}$
$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

# Examples

We will consider:

- $\boldsymbol{\sigma}$-$\mathbf{Set}$;
- $\boldsymbol{\sigma}$-$\mathbf{Gr}$;
- $\boldsymbol{\sigma}$-$\mathbf{Ab}$;
- $\boldsymbol{\sigma}$-$\mathbf{Rng}$.

Given $R \in \boldsymbol{\sigma}$-$\mathbf{Rng}$, consider

- $R$-$\mathbf{Mod}$, the category of difference $R$-modules.

# The topos of difference sets

Let

$$\underline{\mathbb{N}}$$

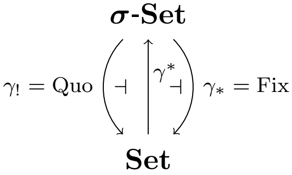
be the category associated with the monoid $(\mathbb{N}, +)$.
Note

$$\boldsymbol{\sigma}\textbf{-Set} \simeq [\underline{\mathbb{N}}, \mathbf{Set}] \simeq \mathbf{B}\mathbb{N}$$

is a Grothendieck topos (as the presheaf category on $\underline{\mathbb{N}}^{\mathsf{op}} \simeq \underline{\mathbb{N}}$), the classifying topos of $\mathbb{N}$.

# The topos of difference sets

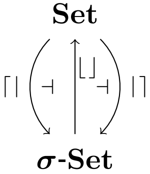
Canonical geometric morphism $\gamma : \boldsymbol{\sigma}\text{-}\mathbf{Set} \to \mathbf{Set}$




The forgetful functor

$$\lfloor\ \rfloor : \boldsymbol{\sigma}\text{-}\mathbf{Set} \to \mathbf{Set}.$$

is the inverse image of an essential point $\mathbf{Set} \to \boldsymbol{\sigma}\text{-}\mathbf{Set}$



# Topos theory view of difference algebra

Old adage of topos theory:

> *A topos can serve as a universe for developing mathematics.*

**Motto**

Difference algebra is the study of algebraic objects internal in the topos $\boldsymbol{\sigma}\text{-}\mathbf{Set}$.

We study ordinary algebra in the internal logic of $\boldsymbol{\sigma}\text{-}\mathbf{Set}$, and then externalise to observe what actually happened.

# Topos theory view of difference algebra

Indeed,

$$\boldsymbol{\sigma}\text{-}\mathbf{Gr} \simeq \mathbf{Gr}(\boldsymbol{\sigma}\text{-}\mathbf{Set})$$
$$\boldsymbol{\sigma}\text{-}\mathbf{Ab} \simeq \mathbf{Ab}(\boldsymbol{\sigma}\text{-}\mathbf{Set})$$
$$\boldsymbol{\sigma}\text{-}\mathbf{Rng} \simeq \mathbf{Rng}(\boldsymbol{\sigma}\text{-}\mathbf{Set}).$$

For $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$,

$$R\text{-}\mathbf{Mod} \simeq \mathbf{Mod}(\boldsymbol{\sigma}\text{-}\mathbf{Set}, R)$$

is the category of modules in a ringed topos.

# Difference homological algebra

Let $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$. Then $R\text{-}\mathbf{Mod}$ is monoidal closed

$$\mathrm{Hom}_R(A \otimes B, C) \simeq \mathrm{Hom}_R(A, [B, C]_R).$$

**Fact (in any Grothendieck topos)**

$R\text{-}\mathbf{Mod} = \mathbf{Mod}(\boldsymbol{\sigma}\text{-}\mathbf{Set}, R)$ is abelian with enough injectives and enough internal injectives.

Cohomology of difference modules is an instance of topos cohomology; we have

- $\mathrm{Ext}^i_R(M, N)$;
- $H^i(\boldsymbol{\sigma}\text{-}\mathbf{Set}, M) = R^i\gamma_*(M) = \mathrm{Ext}^i_R(R, N)$.

# Ext of difference modules

Let $R \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$, and $M, N \in R\text{-}\mathbf{Mod}$, with $M$ étale. Then

$$\operatorname{Ext}^i_R(M,N) = \begin{cases} R\text{-}\mathbf{Mod}(M,N), & i = 0, \\ [M,N]_\sigma, & i = 1, \\ 0, & i > 1, \end{cases}$$

In particular,

$$R^i \operatorname{Fix}(N) = \begin{cases} \operatorname{Fix}(N) = N^\sigma & i = 0, \\ N/\operatorname{Im}(\sigma - \mathrm{id}) = N_\sigma, & i = 1, \\ 0, & i > 1. \end{cases}$$

# Difference algebraic geometry

## Motto

Difference algebraic geometry is algebraic geometry over the base topos $\boldsymbol{\sigma}\text{-}\mathbf{Set}$.

## Plan

1. Pursue the development of algebraic geometry over a base topos $\mathscr{S}$:
   - start with Hakim's monograph;
   - extend to relative étale cohomology and fundamental group(oid).
2. Specialise to $\mathscr{S} = \boldsymbol{\sigma}\text{-}\mathbf{Set}$.

# Hakim spectra

## Hakim-Cole Zariski spectrum

There is a 2-adjunction

$$\begin{array}{c} \mathfrak{Top}.\mathfrak{LocRng} \\ i \Big\downarrow \;\dashv\; \Big\uparrow \mathrm{Spec.Zar} \\ \mathfrak{Top}.\mathfrak{Rng} \end{array}$$

For a ringed topos $(\mathscr{S}, A)$,

$$\mathrm{Spec.Zar}(\mathscr{S}, A)$$

is a locally ringed topos equipped with a morphism of ringed topoi

$$\mathrm{Spec.Zar}(\mathscr{S}, A) \xrightarrow{\pi_{\mathrm{Zar}}} (\mathscr{S}, A).$$

# Hakim spectra

For any 'scheme topology' $\tau$ (Zariski, étale, fppf, fpqc etc), using Hakim's techniques, we can define the $\tau$-spectrum

$$\mathrm{Spec}\,.\tau(\mathscr{S}, A) \xrightarrow{\pi_\tau} (\mathscr{S}, A).$$

## Construction sketch, in terms of Caramello-Zanfa

If $\mathscr{S} = \mathrm{Sh}(\mathscr{C}, J)$,

$$\mathrm{Spec}\,.\tau(\mathscr{S}, A) = \mathrm{Sh}((\mathscr{C}, J) \rtimes ([\mathbf{Sch}_A, \tau])),$$

where the relative site is indexed over $(\mathscr{C}, J)$ by

$$[\mathbf{Sch}_A, \tau]^U = (\mathbf{Sch}_{A(U)}, \tau).$$

# Tierney's internal construction

## Zariski Spectrum via internal sites

Joyal/Tierney's internal frame of radical ideals gives an internal site in $\mathscr{S} = \mathrm{Sh}(\mathscr{C}, J)$

$$(\mathbb{A}, \mathbb{J})$$

so that

$$\mathrm{Spec.Zar}(\mathscr{S}, A) = \mathrm{Sh}_{\mathscr{S}}(\mathbb{A}, \mathbb{J}) \simeq \mathrm{Sh}((\mathscr{C}, J) \rtimes (\mathbb{A}, \mathbb{J})).$$

# Relative schemes

Let $(\mathscr{S}, k)$ be a ringed topos.
We define

$$\mathrm{Sch}_{(\mathscr{S},k)}$$

as the stack completion of

$$U \mapsto \mathrm{Sch}_{\Gamma(U,k)}.$$

Hakim considers its fibre over 1 as the category of $\mathscr{S}$-schemes over $k$.

# Cohomology of $\mathscr{S}$-schemes

Given an $\mathscr{S}$-scheme $X$ and a scheme topology $\tau$, we have a $\tau$-realisation of $X$,

$$
\begin{array}{ccc}
X_\tau & \xrightarrow{\pi_\tau} & \mathscr{S} \\
 & \searrow^{\gamma_\tau} & \downarrow{\scriptstyle \gamma} \\
 & & \mathbf{Set}
\end{array}
$$

If $M$ is an $\mathscr{O}_\tau$-module or an abelian group in $X_\tau$, we define the $\tau$-cohomology and the relative $\tau$-cohomology groups as the topos cohomology groups

$$H^i(X_\tau, M) = R^i\gamma_{\tau,*}(M), \quad \text{and} \quad H^i(X_\tau/\mathscr{S}, M) = R^i\pi_{\tau,*}(M),$$

the latter being abelian groups in $\mathscr{S}$.

# Cohomology of quasi-coherent sheaves

Let $A \in \mathbf{Rng}(\mathscr{S})$, $M \in A\text{-}\mathbf{Mod}$, let $(X, \mathscr{O}_X) = \text{Spec.Zar}(\mathscr{S}, A)$ and

$$\tilde{M} = M \otimes_A \mathscr{O}_X .$$

Hakim: for $i > 0$,

$$H^i(X/\mathscr{S}, \tilde{M}) = 0.$$

# Étale cohomology of relative schemes

## Kummer theory

If $n$ is invertible in an $\mathscr{S}$-scheme $X$, we have a short exact sequence in $X_{\text{ét}}$

$$1 \to \mu_n \to \mathbb{G}_m \xrightarrow{(\,)^n} \mathbb{G}_m \to 1,$$

and we can study the resulting long exact cohomology sequence.

## Artin-Schreier theory

If $X$ is an $\mathscr{S}$-scheme of characteristic $p > 0$, we have an exact sequence in $X_{\text{ét}}$

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathscr{O}_{\text{ét}} \xrightarrow{F-\mathrm{id}} \mathscr{O}_{\text{ét}} \to 0,$$

where $F - \mathrm{id}$ is associated to a homomorphism of additive groups $f \mapsto f^p - f$.

# Magid-Janelidze Galois theory in $\mathscr{S}$

Let $\mathscr{S}$ be a topos. Consider

$$\mathscr{A} \underset{C}{\overset{S}{\rightleftarrows}} \mathscr{P} \qquad (S \dashv C)$$

where

- $\mathscr{A} = \mathbf{Rng}(\mathscr{S})^{\mathrm{op}}$;
- $\mathscr{P} = \mathbf{StoneLoc}(\mathscr{S})$ (compact zero-dimensional locales).
- $S$ is the Pierce spectrum functor

$$S(A) = \text{the locale with frame } \mathrm{Idl}(\mathrm{Idemp}(A)).$$

# Magid-Janelidze Galois theory in $\mathscr{S}$

## Janelidze's Galois theory

For $X \xrightarrow{f} Y$ of relative Galois descent, we obtain a $\mathscr{P}$-internal groupoid

$$\mathrm{Gal}[f] = \left(S(X \times_Y X) \rightrightarrows S(X)\right),$$

and an equivalence of categories

$$\mathrm{Split}_Y(f) \simeq [\mathrm{Gal}[f], \mathscr{P}],$$

an $\mathscr{S}$-version of Magid's separable Galois theory for commutative rings.

# Étale fundamental groupoid of a relative scheme

Let $X$ be an $\mathscr{S}$-scheme.
Then $X_{\text{proét}}$ is a $\mathscr{P}$-determined topos over $\mathscr{S}$, so

**[Bunge 08]**

given a cover $U$ in $X_{\text{proét}}$, get a locally discrete $\mathscr{S}$-localic groupoid

$$\mathbb{G}_U$$

so that the associated fundamental $\mathscr{P}$-pushout is equivalent to its classifying topos,

$$\mathscr{G}_U(X_{\text{proét}}) \simeq \mathscr{B}(\mathbb{G}_U).$$

# Spectra of a difference ring

Let $A \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$ and $\tau$ a scheme topology. Write

$$S_\tau \hookrightarrow \mathbf{Sch}_{/\operatorname{Spec}(\lfloor A \rfloor)}$$

for the classical $\tau$-site of $\operatorname{Spec}(\lfloor A \rfloor)$, so that

$$\lfloor X \rfloor_\tau = \mathrm{Sh}(S_\tau)$$

is the classical $\tau$-topos of $\operatorname{Spec}(\lfloor A \rfloor)$.
Let

$$\mathfrak{s} : S_\tau \to S_\tau$$

be the base change functor along $\operatorname{Spec}(\sigma_A)$.
The data

$$\mathbb{S}_\tau = (S_\tau, \mathfrak{s})$$

defines an internal site in $\boldsymbol{\sigma}\text{-}\mathbf{Set}$, and the $\tau$-spectrum is the topos of internal sheaves

$$X_\tau = \mathrm{Sh}_{\boldsymbol{\sigma}\text{-}\mathbf{Set}}(\mathbb{S}_\tau).$$

# Difference spectra vs classical spectra

Explicitly,

$$X_\tau \simeq \lfloor X \rfloor_\tau^{\varsigma},$$

the category of $\varsigma$-equivariant sheaves in $\lfloor X \rfloor_\tau$, i.e., those $F \in \lfloor X \rfloor_\tau$ equipped with a morphism

$$F \to F \circ \varsigma.$$

There is a natural geometric morphism

$$X_\tau \xrightarrow{\pi_\tau} \boldsymbol{\sigma}\text{-}\mathbf{Set}.$$

We can recover the difference ring as

$$\pi_{\mathrm{Zar}*}\mathscr{O}_X \simeq A.$$

# Étale cohomology of a difference field

We obtain (Hilbert 90)

$$H^1(k_{\text{ét}}, \mathbb{G}_m) \simeq \mathrm{Pic}(k) \simeq (k^\times)_\sigma,$$

and

$$H^2(k_{\text{ét}}, \mathbb{G}_m) \simeq \mathrm{Br}(k) \simeq \mathrm{Br}(\lfloor k \rfloor)^\sigma.$$

Kummer theory gives

$$0 \to ((k^\times)^\sigma)_n \to H^1(k, \mu_n) \to {}_n((k^\times)_\sigma) \to 0,$$

and

$$0 \to ({}_n(k^\times))_\sigma \to H^1(k, \mu_n) \to ((k^\times)_n)^\sigma \to 0.$$

# Étale fundamental groupoid of a difference scheme

Difficulties:

1. a difference scheme $X$ can be topologically totally disconnected, yet indecomposable;
2. $X$ may not have geometric points;
3. the base topos $\boldsymbol{\sigma}$-$\mathbf{Set}$ is not Boolean.

# Étale fundamental groupoid of a difference scheme

Let $\bar{k}$ be a separable closure of $k \in \boldsymbol{\sigma}\text{-}\mathbf{Rng}$, i.e.,
$\lfloor \bar{k} \rfloor$ is the separable closure of $\lfloor k \rfloor$ in the sense of Magid.

## Fundamental difference groupoid [T-Wibmer 21]

Janelidze's Galois theory applied to $f : \bar{k} \to k$ in $\boldsymbol{\sigma}\text{-}\mathbf{Rng}^{\mathrm{op}}$ gives a groupoid in $\boldsymbol{\sigma}\text{-}\mathbf{Prof}$

$$\pi_1(k, \bar{k}) = \mathrm{Gal}[\bar{k} \to k]$$

which classifies the category of difference locally étale $k$-algebras.

## Forthcoming work

A localic version in the $\mathbf{StoneLoc}(\boldsymbol{\sigma}\text{-}\mathbf{Set})$-determined context.

# Difference Galois theory and dynamics

## Connection to symbolic dynamics [T-Wibmer]

If $k$ is a difference field,

$$G = \pi_1(k, \bar{k})$$

is a difference profinite group, and the Galois correspondence restricts to

$$\{\text{finitely } \boldsymbol{\sigma}\text{-presented ind-étale } k\text{-algebras}\} \simeq \{\text{subshifts of finite type with } G\text{-action}\}$$

$$\text{limit degree} \leftrightsquigarrow \text{entropy.}$$

# Difference-differential algebra

## Iannazzo-T: spectrum of a differential ring

- Explicit site for $\mathbf{Set}[\mathbb{T}_{\text{LocalDifferentialRing}}]$.
- Construction of
$$\mathrm{Spec}(A, \delta)$$
using the locale of radical differential ideals.
- Relation to [Carrà Ferro 90].

## Keigher: differential rings in a topos $\mathscr{E}$

$$\mathbf{DRng}(\mathscr{E}).$$

## Difference-differential rings

$$\delta\text{-}\sigma\text{-}\mathbf{Rng} \simeq \mathbf{DRng}(\boldsymbol{\sigma}\text{-}\mathbf{Set}).$$

# Why pursue this programme?

1. Ariadne's thread.

2. Generalisations: we work over the base topos

$$\mathbf{B}\mathbb{N},$$

but one can replace $\mathbb{N}$ by an arbitrary monoid, group, category etc to obtain the corresponding 'equivariant' geometry.