Toposes of Topological Monoid Actions

Morgan Rogers, Università degli Studi dell’Insubria

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A (right) *action* is a set A equipped with a function $" \cdot " : A \times G \to A$ such that the identity element 1_G "does nothing" and for any $g, h \in G$ and $a \in A$,

$$a \cdot (gh) = (a \cdot g) \cdot h.$$

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Alternatively, if we think of G as a category with one object, an action determines a functor $G^{\mathrm{op}} \to \mathbf{Set}$, and these assemble into a functor category which is a prototypical example of a presheaf topos,

$$[G^{\mathrm{op}}, \mathbf{Set}].$$

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Proposition: V is left exact, closed under subobjects, and has a right adjoint.

Proof: A finite product or subset of continuous actions is still continuous. The adjoint sends a right G-set X to its subset of continuous elements,

$$R(X) = \{\, x \in X \mid \mathrm{fix}_G(x) \text{ is open in } \tau \,\}$$

$\square$

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Proof: V is left and exact and comonadic, ensuring that $\mathbf{Cont}(G, \tau)$ inherits the necessary properties (cf Mac Lane and Moerdijk [3, Theorem V.8.4]).

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Proof: Since $\mathrm{Cont}(G,\tau)$ is an elementary topos, $(V \dashv R)$ is a geometric morphism, which is moreover hyperconnected. Consider the quotients of representable (pre)sheaves in $[G^{\mathrm{op}}, \mathrm{Set}]$:

$$\begin{tikzcd} \mathbf{y}(C) \arrow[d, two heads] & & \mathbf{y}(C') \arrow[d, two heads] \\ S & & S' \end{tikzcd}$$

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Any object X in $\mathrm{Cont}(G,\tau)$ is a union of those quotients which are subobjects of X, and these necessarily lie in $\mathrm{Cont}(G,\tau)$. Since any topos is well-copowered, there is a set of these, so we have a generating set of objects. $\square$

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Proof: The quotients of the (unique) representable in $[G^{\mathrm{op}}, \mathbf{Set}]$ are precisely the atoms of this topos, and they are still atoms in $\mathbf{Cont}(G, \tau)$ since V preserves monos. $\square$

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$$\begin{tikzcd}
\mathbf{Set} \arrow[rr, bend left=50, "-\times G"] \arrow[rr, bend right=50, "\mathrm{Hom}_{\mathbf{Set}}(G,-)"'] & & {[G^{\mathrm{op}}, \mathbf{Set}]} \arrow[ll, "U"'] \arrow[ll, phantom, bend left=25, "\perp"] \arrow[ll, phantom, bend right=25, "\perp"] \arrow[rr, shift right=2, "R"'] \arrow[rr, phantom, "\perp"] & & \mathrm{Cont}(G,\tau) \arrow[ll, shift right=2, "V"']
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Once again, denote by $\mathbf{Cont}(M,\tau)$ the subcategory of $[M^{\mathrm{op}}, \mathbf{Set}]$ on the actions which are continuous with respect to τ. Let V be the full and faithful forgetful functor,

$$[M^{\mathrm{op}}, \mathbf{Set}] \leftarrow \mathbf{Cont}(\mathrm{M},\tau) : V$$

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is a member of τ for every element $p \in M$.

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is a member of τ for every element $p \in M$. Equivalently, this can be expressed at the requirement that the right congruence,

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is open in the product topology $\tau \times \tau$. Thus we define:

$$R(X) = \{x \in X \mid \forall q \in M,\, \mathfrak{r}_{xq} \in \tau \times \tau\}.$$

□

Supercompactly generated toposes

In a general presheaf topos, the quotients of representables are the *supercompact* objects, characterized by the following property:

$$\overbrace{\begin{tikzcd} X_i \arrow[dr, "f_i"'] & \cdots & X_j \arrow[dl, "f_j"] \\ & X & \end{tikzcd}}^{i,j \in I} \text{ jointly epic} \implies \exists i^* \in I, \quad \begin{tikzcd} X_{i^*} \arrow[d, two heads, "f_{i^*}"] \\ X \end{tikzcd}$$

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A topos is said to be ***supercompactly generated*** if it has enough supercompact objects.

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Q: Can we recover τ from the right-hand adjunction?

The powerset action

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Theorem: Let $T \coloneqq VR(P(M))$. Then T is a base of clopen sets for the coarsest topology $\tilde{\tau}$ on M such that $\mathbf{Cont}(M,\tau) \simeq \mathbf{Cont}(M,\tilde{\tau})$.

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Moreover, $(M, \tilde{\tau})$ is a topological monoid (even if (M, τ) wasn't)!

Powder monoids

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As such, we say a topological monoid is a **powder monoid** if it is Hausdorff and equipped with an action topology.

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Corollary: Any topological monoid is discrete-action-Morita-equivalent to a powder monoid.

Powder monoids

Examples:

- any discrete monoid
- the profinite natural numbers
- the profinite integers
- any prodiscrete monoid or group
- any “nearly discrete group”
- the group of automorphisms of the natural numbers, with basis of opens given by the stabilizers of finite subsets

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Corollary: $\mathrm{Cont}(M,\tau)$ is a supercompactly generated Grothendieck topos with a surjective point composed of an essential surjection and a hyperconnected morphism.

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Q: Is every topos with such a point a topos of actions of a topological monoid?

Principal actions

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As such, we recover a canonical site for this topos whose objects are the principal continuous M-sets, which we may index by the open right congruences on (M, τ).

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Any endomorphism of the point of $\mathcal{E}$ is a natural transformation whose values are determined by their values at the principal actions lying in $\mathcal{E}$.

Complete monoids

Theorem: Let $\mathcal{R}_h$ be the partial order of open right congruences of M indexing the principal actions lying in $\mathcal{E}$, where $h = (V \dashv R)$ is the hyperconnected geometric morphism $[M^{\mathrm{op}}, \mathrm{Set}] \to \mathcal{E}$.

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$$L := \varprojlim_{r \in \mathcal{R}_h} U(M/r)$$

with multiplication defined by projected composition from M, and equipped with the evident prodiscrete topology, ρ.

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Corollary: A topos is equivalent to the topos of actions of a topological monoid iff it has a point factoring as an essential surjection followed by a hyperconnected morphism.

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Moreover, there is a canonical monoid homomorphism $u : M \to L$; we say a topological monoid is **complete** if this is an isomorphism.

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The powder monoids are precisely those for which u is injective.

Example: The integers (with addition) equipped with the topology having subgroups $p^k\mathbb{Z}$ as a base of opens forms a powder monoid which is not complete; its completion is the group of p-adic integers.

Further highlights

- The construction $\mathbf{Cont}(-)$ can be extended to a 2-functor on the category of topological monoids, semigroup homomorphisms and conjugations.

- Restricting this to complete monoids, the (surjection, inclusion) and (hyperconnected, localic) factorization systems for geometric morphisms correspond to the (monoid hom, idempotent subsemigroup) and (dense, closed) factorizations at the level of monoids.

- When (L, ρ) is a complete monoid, the hyperconnected morphism $[L^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Cont}(L, \rho)$ is “internally full and faithful on essential geometric morphisms”.

- The categories of powder monoids and complete monoids are (2-)monadic over the (2-)category of topological monoids.

Thank You!

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