Introduction to higher topoi

3: Classifiers and topologies

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The story so far:

- an ∞-topos is an ∞-category $\mathcal{E}$, such that $\exists$

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook, "i"'] \arrow[r, phantom, shift left=2, "\perp"] & \mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S}) \arrow[l, bend right, "\ell"'] \end{tikzcd}$$

  - i accessible
  - $\underline{\ell \text{ lex}}$

- equivalently: presentable ∞-category with universal colimits and descent

Classifying spaces in topology:

Fiber bundle $p: E \to B$ with fiber F : $\left(\exists \text{ open cover } \begin{array}{c} E \times_B U_i \xrightarrow{\cong} F \times U_i \\ \searrow \quad \swarrow \\ U_i \end{array} \right)$

$\rightsquigarrow$ Universal bundle: $EG \times_G F \longrightarrow BG$, $G = \mathrm{Aut}(F)$
with fiber F

top'l group of homeo. of F

$$\left\{ \begin{array}{c} \text{bundles } E \to B \\ \text{with fiber } F \\ \text{up to iso} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{maps } B \to BG \\ \text{up to homotopy} \end{array} \right\}$$

Classifying spaces in homotopy

Map $p: E \to B$ with homotopy fiber F.

$\rightsquigarrow$ Universal "bundle": $EG \times_G F \longrightarrow BG$, $G = hAut(F)$

$$\left\{ \begin{array}{l} \text{maps } E \to B \\ \text{with ho fib} \simeq F \\ \text{up to } \begin{tikzcd} E \arrow[rr, "\text{w.e.}", "\sim"'] \arrow[dr] & & E' \arrow[dl] \\ & B & \end{tikzcd} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{maps} \\ B \longrightarrow BG \\ \text{up to homotopy} \end{array} \right\}$$

$h\mathrm{Aut}(F) \subseteq \mathrm{Map}(F,F)$ subspace of homotopy equivalences.

topological monoid

"grouplike": inverses up to homotopy

Stasheff, Dold-Sugawara (60's)

May "Classifying spaces and fibrations", (1975)

(also sSet analog, J. Moore and others (50-60's).)

Example: $F := K(G,n)$ G abelian group, $n \geq 2$

$\pi_n \cong G$, $\pi_k = 0$ if $k \neq n$

$\Rightarrow$ $hAut(K(G,n)) \simeq Aut(G) \ltimes K(G,n)$

$$\Rightarrow \left\{ \begin{array}{l} E \to B \text{ with} \\ \text{ho fiber } K(G,n) \end{array} \right\} \Longleftrightarrow \left\{ B \to B\,hAut(K(G,n)) \right\}$$

$\Downarrow$ $\pi_1 = Aut(G)$, $\pi_{n+1} = G$

$\parallel$

$n=1$, G group $\Rightarrow$

$B(hAut(K(G,1))) =$ "2-groupoid"

$\Rightarrow$ $\pi_1 = Out(G)$, $\pi_2 = Centr(G)$ $(\pi_k = 0)$

Put all the $B(hAut(F))$ together:

$$\left\{ \begin{array}{c} \text{maps } E \to B \\ \text{(up to } \begin{tikzcd} E \arrow[rr, "\text{we}"] \arrow[dr] & & E' \arrow[dl] \\ & B & \end{tikzcd} \text{)} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{maps} \\ B \longrightarrow \overbrace{\coprod_{[F]} BhAut(F)}^{\Omega} \\ \text{(up to homotopy)} \end{array} \right\}.$$

$$\begin{tikzcd} \Omega_* \arrow[d] \\ \Omega \end{tikzcd}$$

"universal map" $\Omega_* \longrightarrow \Omega$ — "large"

$\rightsquigarrow$ characteristic property of $\mathcal{S}$ $[\, \Omega \simeq \mathcal{S}^{\simeq} \,]$

∞-category

maximal ∞-groupoid $\subseteq \mathcal{S}$

Recall: $\mathrm{Cart}(\mathcal{E}^{\rightarrow}) \subseteq \mathcal{E}^{\rightarrow} = \mathrm{Fun}(\Delta^1, \mathcal{E})$

Dream: $\Omega_* \to \Omega$ terminal object of $\mathrm{Cart}(\mathcal{E}^{\rightarrow})$

Universal family: $U_* \xrightarrow{u} U$ subterminal ("(-1)-truncated") object of $\mathrm{Cart}(\mathcal{E}^{\rightarrow})$

$$\forall\, p \qquad \mathrm{Map}_{\mathrm{Cart}(\mathcal{E}^{\rightarrow})}(p, u) \simeq \begin{cases} * \\ \emptyset \end{cases} \text{ or}$$

Lurie, HTT, Ch 6

Gepner-Koch, "Univalence in locally closed ∞-categories" (arXiv 2017)

Rasekh, "Univalence in higher category theory" (arXiv 2021)

<u>Example</u>: $\mathcal{E}$ = 1-topos $\rightsquigarrow$ subobject classifier

$$1 \rightarrowtail \Omega^{\text{mono}}, \qquad \text{so}$$
universal family

$$\left\{ \begin{array}{l} \text{monomorphisms} \\ E \rightarrowtail B \\ (\text{up to } \begin{tikzcd} E \arrow[rr, "\cong"] \arrow[dr] & & E' \arrow[dl] \\ & B & \end{tikzcd}) \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} \text{morphisms} \\ B \to \Omega^{\text{mono}} \end{array} \right\}.$$

in Set: all univ families are mono

Fact: $u \in \mathrm{Cart}(\mathcal{E}^{\to})$ universal family $\iff$ $\mathrm{Cart}(\mathcal{E}^{\to})_{/u} \rightarrowtail \mathrm{Cart}(\mathcal{E}^{\to})$ is fully faithful

Essential image $\mathcal{L} := \mathcal{L}_u \subseteq \mathrm{Cart}(\mathcal{E}^{\to})$ is a local class:

(1) $f \to f'$ in $\mathrm{Cart}(\mathcal{E}^{\to})$, $f' \in \mathcal{L} \implies f \in \mathcal{L}$ closed under base change

(2) $\mathcal{L}$ has colimits and $\mathcal{L} \to \mathcal{E}^{\to}$ preserves colimits

descent $\iff$ $\mathrm{Cart}(\mathcal{E}^{\to})$ is a local class

$u : U_* \to U$ universal family $\Longrightarrow \mathcal{L} = \mathcal{L}_u$ is a **bounded** local class:

$$\begin{tikzcd}
\left\{ \begin{smallmatrix} X \\ \downarrow \\ B \end{smallmatrix} \in \mathcal{L} \right\} \arrow[r, phantom, "\rightarrow"] & (\mathcal{E}_{/B})^{\simeq}_{\mathcal{L}} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{E}_{/B})^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{B\} \arrow[d] & \text{object of } \mathcal{E} \\
\text{local class} \arrow[r, phantom, "\rightarrow"] & \mathcal{L} \arrow[r, tail] & \mathrm{Cart}(\mathcal{E}^{\rightarrow}) \arrow[r, "\text{target}"] & \mathcal{E} & \text{pb in } \mathrm{Cat}_\infty
\end{tikzcd}$$

$$\begin{smallmatrix} X \\ \downarrow \\ Y \end{smallmatrix} \longmapsto Y$$

bounded means each $(\mathcal{E}_{/B})^{\simeq}_{\mathcal{L}}$ is essentially small ∞-groupoid

$\left[\ (\mathcal{E}_{/B})^{\simeq}_{\mathcal{L}} \simeq \mathrm{Map}_{\mathcal{E}}(B, U) \ \right]$

$\mathcal{E}$ ∞-topos

Prop: Correspondence

$$\left\{\begin{array}{l}\text{universal families} \\ u : U_* \to U \ \text{ in } \mathrm{Cart}(\mathcal{E}^{\to})\end{array}\right\} \rightleftarrows \left\{\begin{array}{l}\text{bounded local classes} \\ \underline{\underline{\mathcal{L}}} \subseteq \mathrm{Cart}(\mathcal{E}^{\to})\end{array}\right\}$$

Idea: $B \longmapsto \widehat{(\mathcal{E}_{/B})_{\mathcal{L}}}$ is a functor $\mathcal{E}^{op} \to \mathcal{S}$

small ∞-gpds

univ colim + descent $\Rightarrow$ preserves limits, $\mathcal{E}$ presentable $\Rightarrow$ representable by U

Prop: Every morphism in an ∞-topos is contained in some bounded local class

Proof idea: $$\mathrm{Cart}(\mathcal{E}^{\to}) = \bigcup \mathcal{L}^{\kappa}$$

$\mathcal{L}^{\kappa}$ = bounded local class of "relatively κ-compact" morphisms

regular cardinal

Lurie, Gepner-Koch, Rasekh

$\rightsquigarrow$ get an exhaustive collection $\{ U^{\kappa}_{*} \to U^{\kappa} \}$ of universal families

$\rightarrow$ the "union" (in a "higher universe"): $\underbrace{\Omega_{*} \to \Omega}_{\text{universal map}}$

Theorem: $\mathcal{E}$ is an ∞-topos iff presentable, univ. colimits, "enough" universal families

Univalence :

$p : E \to B$ in $\mathcal{E}$ $\Longrightarrow$ $\exists$ $\begin{tikzcd} B \arrow[dr, "i"'] & \\ & \mathrm{Iso}(p) \arrow[r, "\pi"] & B \times B \end{tikzcd}$ ← "id."

∞-topos (or 1-topos)

$$\left\{ \begin{tikzcd} & \mathrm{Iso}(p) \arrow[d, "\pi"] \\ T \arrow[ur, dashed] \arrow[r, "(f,g)"'] & B \times B \end{tikzcd} \right\} \Longleftrightarrow \left\{ \begin{tikzcd} f^*E \arrow[rr, dashed, "\mathrm{iso}"] \arrow[dr] & & g^*E \arrow[dl] \\ & T & \end{tikzcd} \right\}$$

$p : E \to B$ is <u>univalent</u> iff $B \xrightarrow[\simeq]{i} \mathrm{Iso}(p)$ is iso

<u>Thm</u>: p univ family $\Longleftrightarrow$ p is univalent.

Van Kampen colimits:

$\mathcal{E}$ ∞-category with pullbacks

$f: A \to B$ in $\mathcal{E}$ $\rightsquigarrow$ $f^*: \mathcal{E}_{/B} \to \mathcal{E}_{/A}$

A colimit cone $F: J \star \Delta^0 \to \mathcal{E}$ is a van Kampen colimit if induced functor

$$(J \star \Delta^0)^{op} \longrightarrow \widehat{Cat}_\infty, \quad j \longmapsto \mathcal{E}_{/F(j)}$$

is a limit cone.

Example: In a 1-topos $\mathcal{E}$, coproducts are van Kampen

i.e, if $X \simeq \coprod X_i, \in \mathcal{E}$

$$\mathcal{E}_{/X} \xrightarrow{\simeq} \prod_i \mathcal{E}_{/X_i} \checkmark$$

in General, pushouts are not van Kampen in a 1-topos

Theorem: $\mathcal{E}$ is an ∞-topos iff presentable, all colimits are van Kampen

Proof idea: if $X \simeq \operatorname{colim}_{i \in I} X_i \in \mathcal{E}$

$$\begin{tikzcd} \mathcal{E}_{/X} \arrow[r, shift right, "\text{pull back}"'] & \lim_i \mathcal{E}_{/X_i} \arrow[l, shift right, "\text{colim}"'] \end{tikzcd} \ni \left\{ \begin{matrix} E_i \\ \downarrow \\ X_i \end{matrix} \right\}_i$$

$\circlearrowright = \mathrm{id}$ univ colim

$\circlearrowleft = \mathrm{id}$ descent

Have $\mathcal{E}^{op} \to \widehat{\mathrm{Cat}}_\infty$, $A \mapsto \mathcal{E}_{/A}$

For each bounded local class $\mathcal{L}$, have

$$\mathcal{E}^{op} \to \mathrm{Cat}_\infty, \quad A \mapsto (\mathcal{E}_{/A})_{\mathcal{L}} \subseteq \mathcal{E}_{/A}$$

$\leftarrow$ small ∞-category

functor preserves limits, $\mathcal{E}$ presentable

$\Rightarrow$ representable by an internal ∞-category object $U_\bullet^{\mathcal{L}}$ of $\mathcal{E}$

Theorem: $\mathcal{E}$ is an ∞-topos iff presentable, every morphism in a local class represented by internal ∞-category object

Rasekh, "A theory of elementary higher toposes" (arXiv 2018)

Elementary ∞-topoi ?

What does "elementary" mean?

Definitions by Shulman, Rasekh.

- finite complete and cocomplete
- has subobject classifier.
- every morphism in a local class represented by internal ∞-category object.

$\Longrightarrow$ loc cart closed

Monomorphism: $f: A \to B$ such that $\Delta(f): A \to A \times_B A$ iso

Orthogonal maps: $f \perp g$ if

unique lift exists in any

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd} \iff \begin{tikzcd} \mathrm{Map}(B,X) \arrow[r, "g_*"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}(B,Y) \arrow[d, "f^*"] \\ \mathrm{Map}(A,X) \arrow[r, "g_*"'] & \mathrm{Map}(A,Y) \end{tikzcd}$$

(up to "contractible choice")

Cover: $f: A \to B$ such that $f \perp g$ $\forall$ $g \in \underline{\mathrm{Mono}}$

(or "effective epimorphism")
(or "surjection") usually not epimorphisms!

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dotted, "\exists !"] & Y \end{tikzcd}$$

Example: In $\mathcal{S}$ = ∞-groupoids, (= htpy of spaces)

$f: A \to B$ cover iff $\pi_0 A \to \pi_0 B$ surjective

sets of path components

"lift points up to homotopy"

Observation: Mono is stable under pullback. So

$f : A \to B$ cover iff ~~unique~~ lift exists in every

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & U \arrow[d, tail, "g \in \mathrm{Mono}"] \\ B \arrow[r, equal] \arrow[ur, dashed, "\exists"] & B \end{tikzcd} \iff \begin{array}{l} \forall\ g : U \rightarrowtail B \\ \mathrm{Map}_{\mathcal{E}/B}(f, g) \neq \emptyset \\ \text{only if } g \text{ iso} \end{array}$$

$\mathrm{Map}_{\mathcal{E}/B}(h, g) \in \begin{cases} \emptyset \\ * \end{cases}$

$\Rightarrow$ f cover in $\mathcal{E}$ iff cover in $\mathcal{E}/B$

$\Rightarrow$ gf cover $\Rightarrow$ g cover.

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[dr, "gf"'] & \cdot \arrow[d, "g"] \\ & \cdot \end{tikzcd} \Leftarrow$$

Consequence: If $F: I \to \mathcal{E}$ has colimit B,

then $\coprod_{i \in I_0} F(i) \twoheadrightarrow B$ is a cover.

Proof: WLOG $B = 1$. Let $U \rightarrowtail 1$. Then

$$\begin{tikzcd} \coprod F(i) \arrow[r] \arrow[d] & U \arrow[d, tail] \\ 1 \arrow[r, equal] \arrow[ur, dashed, "?"] & 1 \end{tikzcd} \quad \Rightarrow \quad \begin{aligned} \mathrm{Map}(1, U) &\simeq \mathrm{Map}(\mathrm{colim}_i F(i), U) \\ &\simeq \lim_i \mathrm{Map}(F(i), U) \end{aligned}$$

In a presentable ∞-category, (Cover, Mono) is a

factorization system:

- Cover $\perp$ Mono
- Cover, Mono stable under retracts.
- Every f admits $f = i \circ p$. $p \in$ Cover, $i \in$ Mono

$f : A \to B$

essentially unique:

$$A \xrightarrow{p} \mathrm{Im}(f) \xrightarrow{i} B$$

In an ∞-topos, construct cover/mono factorization

Čech nerve: $f: A \to B \quad \rightsquigarrow \quad C_f : \Delta_+^{op} \to \mathcal{E}$

$\Delta_+^{op} = \Delta^{op} * \Delta^0$ aug. simplicial object

$C_f : \cdots \; A \times_B A \times_B A \;\substack{\to\\ \leftarrow\\ \to\\ \leftarrow\\ \to}\; A \times_B A \;\substack{\to\\ \leftarrow\\ \to}\; A \xrightarrow{f} B$

$\rightsquigarrow \quad A \overset{p}{\twoheadrightarrow} \operatorname{colim}_{\Delta^{op}} C_f \overset{i}{\rightarrowtail} B$

$\operatorname{colim}_{\Delta^{op}} C_f = \operatorname{Im}(f)$

Proof: WLOG, $B=1$ ($\mathcal{E} \rightsquigarrow \mathcal{E}_{/B}$) $\quad f : A \to 1$

$$C_f: \quad \cdots \; A\times A\times A \;\substack{\to\\\to\\\to}\; A\times A \rightrightarrows A \to 1$$

$$A\times C_f: \quad \cdots \; A\times A\times A\times A \;\substack{\to\\\to\\\to}\; A\times A\times A \rightrightarrows A\times A \to A$$

"contracting homotopy" $\left(\Delta_+^{op} \to \Delta_0^{op},\ \to \mathcal{E} \right)$.

"absolute colimit" every composite $\Delta_+^{op} \to \Delta_0^{op} \to \mathcal{C}$ is a colimit cone in $\mathcal{C}$.

So (1) $A\times E = \mathrm{colim}_{\Delta^{op}} A\times C_f \xrightarrow{\sim} A$ $\qquad E=\mathrm{colim}\, C_f$

univ colim $\qquad \mathrm{colim}\, \underline{A}^{n\times 1}$

$\Rightarrow$ (2) $E\times E \simeq \mathrm{colim}_{\Delta^{op}} C_f\times E \xrightarrow{\simeq} \mathrm{colim}_{\Delta^{op}} C_f \simeq E$ ✓

So $E \xrightarrow{\simeq} E\times E \Rightarrow \underline{E \rightarrowtail 1}$

(3) $E\simeq \mathrm{colim}_{\Delta^{op}} C_f \Rightarrow$

$$\begin{tikzcd}
\coprod_{n\geq 0} A^{\times n+1} \arrow[rr, two heads, "\text{cover}"] \arrow[dr] & & E \\
& A \arrow[ur, "p"'] &
\end{tikzcd}$$

✓ $\Rightarrow p$ cover.

Covers and local classes: $\mathcal{L}$ is a local class iff $\subseteq \mathrm{Cart}(\mathcal{E}^{\to})$

(1) $\mathcal{L}$ closed under coproducts

and $\forall$
$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\ A \arrow[r, "p"'] & B \end{tikzcd}$$
pullback.

(2) $g \in \mathcal{L} \implies f \in \mathcal{L}$

(3) if $p \in \mathrm{Cover}$, $f \in \mathcal{L} \implies g \in \mathcal{L}$

Consequence: given $f : E \to B$, let $\in \mathcal{E}$

$$\mathcal{L}^f := \left\{ g : E'' \to B'' \text{ st } \exists \begin{tikzcd} E & E' \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\llcorner" very near start] \arrow[dr, phantom, "\lrcorner" very near start] & E'' \arrow[d, "g"] \\ B \arrow[u, leftarrow, "f"'] & B' \arrow[l] \arrow[r, two heads, "\text{cover}"'] & B'' \end{tikzcd} \right\}$$

"g looks locally like f"

$\Rightarrow$ $\mathcal{L}$ is the smallest local class containing f, (bounded)

$\Rightarrow$ $\exists$ $U^f_* \to U^f$ univ. family

Proof of cover characterization of local class:

($\Rightarrow$): local class $\Rightarrow$ (1) $\amalg$ closed, (2) base change

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ T $\qquad\qquad$ ✓ $\qquad\qquad$ T

· b dom cov

$\overset{?}{g} \in \mathcal{L}$, $f \in \mathcal{C}$

(3) pull back along cover:

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X \times_Y X \times_Y X \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, "f_2"] \arrow[dr, phantom, very near start, "\lrcorner"] & X \times_Y X \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "f_1"] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[r, two heads] \arrow[d, "f"] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "g"] & \leftarrow \text{colim (univ colim)} \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A \times_B A \times_B A \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & A \times_B A \arrow[r, shift left] \arrow[r, shift right] & A \arrow[r, two heads, "p"'] & B & - \text{colim}
\end{tikzcd}$$

$\Rightarrow f_k \in \mathcal{L} \Rightarrow g \in \mathcal{L}$.

Čech nerve $\qquad\qquad$ cover

Proof (ct'd) ($\Leftarrow$): $(1), (2), (3) \overset{?}{\Longrightarrow} \mathcal{L}$ closed under colims in $\mathrm{Cart}(\mathcal{E}^{\to})$

$\underline{\Rightarrow \text{ coproducts}}$

$\gamma : G \to F$ Cartesian nat transf of $I \to \mathcal{E}$

each $G(i) \xrightarrow{\varphi(i)} F(i) \in \mathcal{L}$ $\quad (I \to \mathcal{L} \subseteq \mathrm{Cart}(\mathcal{E}^{\to}))$

$\mathrm{colim}_I \Downarrow$

$g : Y \to B \Longrightarrow$

$$\begin{tikzcd}
\coprod_i G(i) \arrow[r] \arrow[d, "\coprod_i \varphi(i)"'] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d, "g"] \\
\coprod_i F(i) \arrow[r, "p"'] & B
\end{tikzcd}$$

cover

$\Leftarrow$ descent:

$$\begin{tikzcd}
G(i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y \arrow[d] \\
F(i) \arrow[r] & B
\end{tikzcd}$$

pb.

$\Longrightarrow g \in \mathcal{L}.$

These ideas also show:

Mono and Cover are local classes in $\mathrm{Cart}(\mathcal{E}^{\to})$

<u>Proof</u>:

	<u>Mono</u>	<u>Cover</u>
(1) $\coprod$ stable :	descent	$\perp$ to Mono
(2) base change :	easy	descent + Čech nerve
(3) p.b along cover :	descent + Čech nerve	closed under composition, $gf \in \mathrm{Cover} \Rightarrow g \in \mathrm{Cover}$.

Truncation:

In an ∞-category $\mathcal{C}$

- $f: A \to B$ is n-truncated if $\Delta^{n+2}(f): A \to \Omega^{n+2}(f)$ is iso
- X n-truncated object $\Leftrightarrow$ $X \to 1$ is n-truncated.
- $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$ full subcategory of n-truncated objects

$B \in \mathcal{C}_{\leq n} \Rightarrow \mathrm{Map}_{\mathcal{C}}(A, B) \in \mathcal{S}_{\leq n}$ so $\mathcal{C}_{\leq n}$ is an "$(n+1)$-category"

"n-groupoid"

"
"$(n+1, 1)$-category"

$$\mathrm{Iso} \subseteq \mathrm{Mono} \subseteq \mathrm{Trunc}_0 \subseteq \mathrm{Trunc}_1 \subseteq \mathrm{Trunc}_2 \subseteq \cdots \subseteq \mathcal{C}^{\to}$$

$$\Downarrow \perp$$

$$\mathcal{C}^{\to} \supseteq \mathrm{Cover} \supseteq \mathrm{Conn}_1 \supseteq \mathrm{Conn}_2 \supseteq \mathrm{Conn}_3 \supseteq \cdots \supseteq \mathrm{Iso}$$

where $f: A \to B$ is **n-connective** if $f \perp \mathrm{Trunc}_{n-1}$

(also "n-connected", also "(n-1)-connected")

object X is **n-connective** if $(X \to 1) \in \mathrm{Conn}_n$

Properties: (in presentable ∞-category)

- $(\mathrm{Conn}_{n+1}, \mathrm{Trunc}_n)$ is a factorization system

$A \xrightarrow{f} B \Rightarrow A \xrightarrow{(n+1)\text{-conn}} \mathrm{Im}_n(f) \xrightarrow{n\text{-trunc}} B$ relative n-truncation

$X \Rightarrow X \xrightarrow{(n+1)\text{-conn}} \mathrm{Tr}_n(X) \xrightarrow{n\text{-trunc}} 1$ absolute n-truncation

- Conn_n, Trunc_n are local classes

$\Rightarrow X \to \cdots \to \mathrm{Tr}_{n+1}(X) \to \mathrm{Tr}_n(X) \to \mathrm{Tr}_{n-1}(X)$

Example: In $\mathcal{S}$:

- X n-truncated iff $\pi_k(X,x)=0 \quad \forall\, x\in X,\ k>n$
- X n-connective iff $\pi_k(X,x)=0 \quad \forall\, x\in X,\ k\leq n$
  $(n\geq 0)$ and $X\neq\emptyset$ "$(n-1)$-connected space"
- $f: X\to Y$ is $\begin{cases} \text{n-truncated} \\ \text{n-connective} \end{cases}$: $\longrightarrow$ "n-connected map"

  iff all its htpy fibers are $\begin{cases} \text{n-trunc} \\ \text{n-conn} \end{cases}$

In $\mathcal{S}$, $X \to Tr_n X$ is constructed by

"killing homotopy in dim $>n$" by

"attaching cells in dims $\geq n+2$"

In $PSh(C)$, $Tr_n X$ computed "pointwise"

In $\mathcal{E} \underset{i}{\overset{\ell \text{ lex}}{\rightleftarrows}} PSh(C)$, $Tr_n^{\mathcal{E}}(X) \simeq \ell\left(Tr_n^{PSh(C)}(iX)\right)$

ℓ, i preserve "n-truncation" property

Example:

n-gerbes: $\quad \mathrm{Gerbe}_n := \mathrm{Trunc}_n \cap \mathrm{Conn}_n.$

(= "EM n-gerbes") $\quad (\mathrm{Trunc}_n \cap \mathrm{Conn}_{n+1} = \mathrm{Iso})$

$\mathcal{E}_{\mathrm{Gerbe}_n} \subseteq \mathcal{E}$: full subcategory spanned by

∞-topos

$(E \xrightarrow{p} 1) \in \mathrm{Gerbe}_n$

$(\mathcal{E}_{\mathrm{Gerbe}_n})_* = \{ 1 \xrightarrow{s} E \xrightarrow{p} 1, \; p \in \mathrm{Gerbe}_n \}$. $\leftarrow$ is a 1-category.

Prop: $(\mathcal{E}_{\text{Gerben}})_* \xrightarrow{\sim} (\mathcal{E}_{\leq 0})_{ab}$ ← abelian group objects in 1-category $\mathcal{E}_{\leq 0}$ ← (1-topos)

$(n \geq 2)$

1-category

($n=1$, group objects, $n=0$, pointed objects)

Construction of $(\mathcal{E}_{\text{Gerben}})_* \xrightarrow{\Omega^n} \mathcal{E}_{\leq 0}$:

$$\left(\underbrace{1 \xrightarrow{s} E \xrightarrow{p} 1}\right) \Rightarrow \Delta^n(s) : 1 \longrightarrow \Omega^n(s), \; \Omega^n(s) \in \mathcal{E}_{\leq 0}$$

Inverse: $A \longmapsto K(A, n) \in (\mathcal{E}_{\text{Gerben}})_*$

Example: Given $A \in (\mathcal{E}_{\leq 0})_{ab} \Rightarrow (K(A,n) \xrightarrow{p} 1) \in \mathcal{E}$

$\rightsquigarrow$ universal family $U^{A,n}_* \to U^{A,n}$ of gerbes "locally like p"

<u>$n \geq 2$</u>: pullbacks

$$\begin{tikzcd}
K(A,n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner", very near start] & U^{A,n}_* \arrow[d] \\
1 \arrow[r] & K(A,n+1) \arrow[r] & U^{A,n}.
\end{tikzcd}$$

p is not a univ family

classifies "gerbes banded by A"

$\mathrm{Conn}_\infty := \bigcap \mathrm{Conn}_n$ ∞-connected maps

$\Leftrightarrow$ $f: A \to B$ st $\mathrm{Im}_n(f) \simeq B$ $\forall n$

also a local class

X ∞-connected object if $(X \to 1) \in \mathrm{Conn}_\infty$.

Example: In $\mathcal{S}$, $\mathrm{PSh}(C)$, $\mathrm{Conn}_\infty = \mathrm{Iso}$

"Whitehead thm"

Example: Non-trivial ∞-connected object in ∞-topos

$$
\begin{tikzcd}
 & U_0^+ \arrow[dl] & & U_1^+ \arrow[dl] & & U_2^+ \arrow[dl] & & \arrow[dl] & \\
X = V_0 & & V_1 \arrow[ul] \arrow[dl] & & V_2 \arrow[ul] \arrow[dl] & & V_3 \arrow[ul] \arrow[dl] & & \cdots \\
 & U_0^- \arrow[ul] & & U_1^- \arrow[ul] & & U_2^- \arrow[ul] & & \arrow[ul] &
\end{tikzcd}
$$

top'l space $\qquad\qquad\qquad\qquad$ $\leftarrow \mathrm{Open}_X$

$F : \mathrm{Open}_X^{op} \longrightarrow \mathcal{S}$ so that $\begin{cases} F(U_n^\pm) \simeq * \\ F(V_n) \simeq K(\mathbb{Z}, n) \end{cases}$

Eilenberg-MacLane space

$F:$

$$\begin{tikzcd}
 & 1 \arrow[dr] & & 1 \arrow[dr] & & 1 \arrow[dr] & & \cdots \\
K(\mathbb{Z},0) \arrow[ur] \arrow[dr] \arrow[rr, phantom, ">"] & & K(\mathbb{Z},1) \arrow[ur] \arrow[dr] \arrow[rr, phantom, ">"] & & K(\mathbb{Z},2) \arrow[ur] \arrow[dr] \arrow[rr, phantom, ">"] & & K(\mathbb{Z},3) \arrow[ur] \arrow[dr] & \\
 & 1 \arrow[ur] & & 1 \arrow[ur] & & 1 \arrow[ur] & & \cdots
\end{tikzcd}$$

$K(\mathbb{Z},n) \simeq \Omega K(\mathbb{Z},n+1)$

so $\underbrace{F \in Sh(X)}_{\infty\text{-topos}} \underset{\hookrightarrow}{\overset{\ell}{\leftarrow}} Fun(Open_X^{op}, \mathcal{S})$

F is ∞-connected, $F \not\simeq *$

$$\left\{ \begin{array}{l} Tr_m^{sh} F = \ell(Tr_m^{psh} F) \\ Tr_m(K(\mathbb{Z},n)) = * \quad \text{if } n > m \end{array} \right.$$

Next time:

- Grothendieck topologies?
- Geometric morphisms