

# Introduction to higher topoi

## 3: Classifiers and topologies

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## The story so far:

- an  $\infty$ -topos is an  $\infty$ -category  $\mathcal{E}$ , such that  $\exists$

$$\mathcal{E} \begin{array}{c} \xleftarrow{\ell} \\ \perp \\ \xrightarrow{i} \end{array} \text{PSh}(\mathcal{C}) := \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{S})$$

•  $i$  accessible  
•  $\ell$  lex

- equivalently: presentable  $\infty$ -category with universal colimits and descent

# Classifying spaces in topology:

Fiber bundle  $p: E \rightarrow B$  with fiber  $F$  :  $\left( \begin{array}{c} \exists \\ \text{open} \\ \text{cover.} \end{array} \quad \begin{array}{c} E \times U_i \xrightarrow{\cong} F \times U_i \\ \searrow \quad \swarrow \\ \quad U_i \end{array} \right)$

↪ Universal bundle:  
with fiber  $F$

$$EG \times_G F \rightarrow BG,$$

$G = \text{Aut}(F)$   
top'l group of homeo.  
of  $F$

{ bundles  $E \rightarrow B$   
with fiber  $F$   
up to iso }  
↑



{ maps  $B \rightarrow BG$   
up to homotopy }  
↑

# Classifying spaces in homotopy

Map  $p: E \rightarrow B$  with homotopy fiber  $F$ .

↪ Universal "bundle":  $EG \times_G F \rightarrow BG, G = \text{hAut}(F)$

{ maps  $E \rightarrow B$   
with ho fib  $\simeq F$   
up to  $E \xrightarrow{\sim} E'$   
           $\downarrow$   
           $B$  }

$\iff$

{ maps  $B \rightarrow BG$   
up to homotopy }

$$\mathbf{hAut}(F) \subseteq \mathbf{Map}(F, F)$$

topological monoid

"grouplike": inverses up to homotopy

subspace of homotopy  
equivalences.

Stasheff, Dold-Sugawara (60's)

May. "Classifying spaces and fibrations", (1975) ←

(also sSet analog, J. Moore and others (50-60's).)

Example:  $F := K(G, n)$        $G$  abelian group,  $n \geq 2$   
 $\pi_n = G$ ,  $\pi_k = 0$  if  $k \neq n$

$$\Rightarrow \text{hAut}(K(G, n)) \simeq \underline{\text{Aut}(G)} \times K(G, n)$$

$$\Rightarrow \left\{ \begin{array}{l} E \rightarrow B \text{ with} \\ \text{no fiber } K(G, n) \end{array} \right\} \iff \left\{ B \rightarrow B \text{ hAut}(K(G, n)) \right\}$$

$$\Downarrow \pi_1 = \text{Aut}(G)$$

$$\pi_{n+1} = G$$

$$\left[ \begin{array}{l} n=1, G \text{ group} \Rightarrow \\ B(\text{hAut}(K(G, 1))) = \text{"2-groupoid"} \\ \Rightarrow \left( \begin{array}{l} \pi_1 = \text{Out}(G) \\ \pi_2 = \text{Center}(G) \end{array} \right) (\pi_k = 0) \end{array} \right]$$

Put all the  $B(\text{hAut}(F))$  together:

$$\left\{ \begin{array}{l} \text{maps } E \rightarrow B \\ \text{(up to } E \xrightarrow{u} E' \\ \quad \searrow \quad \nearrow \\ \quad B \end{array} \right\} \iff \left\{ \begin{array}{l} \text{maps} \\ B \rightarrow \coprod_{[F]} B(\text{hAut}(F)) \\ \text{(up to homotopy)} \end{array} \right\}$$

"universal map"

$$\Omega_* \rightarrow \Omega \quad \text{"large"}$$



characteristic property of  $\mathcal{S}$   $\left[ \Omega \simeq \mathcal{S}^{-1} \right]$

$\nearrow$   
 $\infty$ -category

$\nearrow$   
 maximal  
 $\infty$ -group  $\in \mathcal{S}$

Recall:  $\text{Cart}(\mathcal{E}^{\rightarrow}) \subseteq \mathcal{E}^{\rightarrow} = \text{Fun}(\Delta^1, \mathcal{E})$

Dream:  $\Omega_* \rightarrow \Omega$  terminal object of  $\text{Cart}(\mathcal{E}^{\rightarrow})$

Universal family:  $U_* \xrightarrow{u} U$  subterminal ("(-1)-truncated")  
object of  $\text{Cart}(\mathcal{E}^{\rightarrow})$

$$\forall p \quad \text{Map}_{\text{Cart}(\mathcal{E}^{\rightarrow})}(p, u) \simeq \begin{cases} * \\ \emptyset \end{cases} \text{ or}$$

Lurie, HTT, Ch 6

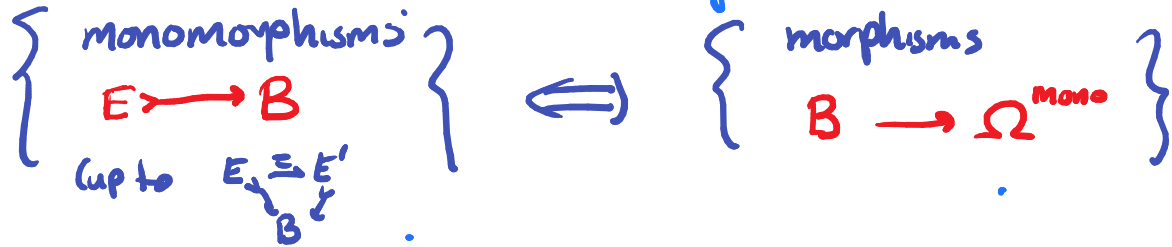
Gepner-Koch, "Univalence in locally closed  $\omega$ -categories" (arXiv 2017)

Rasekh, "Univalence in higher category theory" (arXiv 2021)



Example:  $\mathcal{E} = \mathbf{1}\text{-topos} \rightsquigarrow$  subobject classifier

$1 \xrightarrow{\text{univ. family}} \Omega^{\text{mono}}$ , so



in Set: all univ families are mono

Fact:  $u \in \text{Cart}(\mathcal{E} \rightarrow \Sigma)$   
universal family  $\iff \text{Cart}(\mathcal{E} \rightarrow \Sigma)_u \rightarrow \text{Cart}(\mathcal{E} \rightarrow \Sigma)$   
 $\xrightarrow{\text{is fully faithful}}$

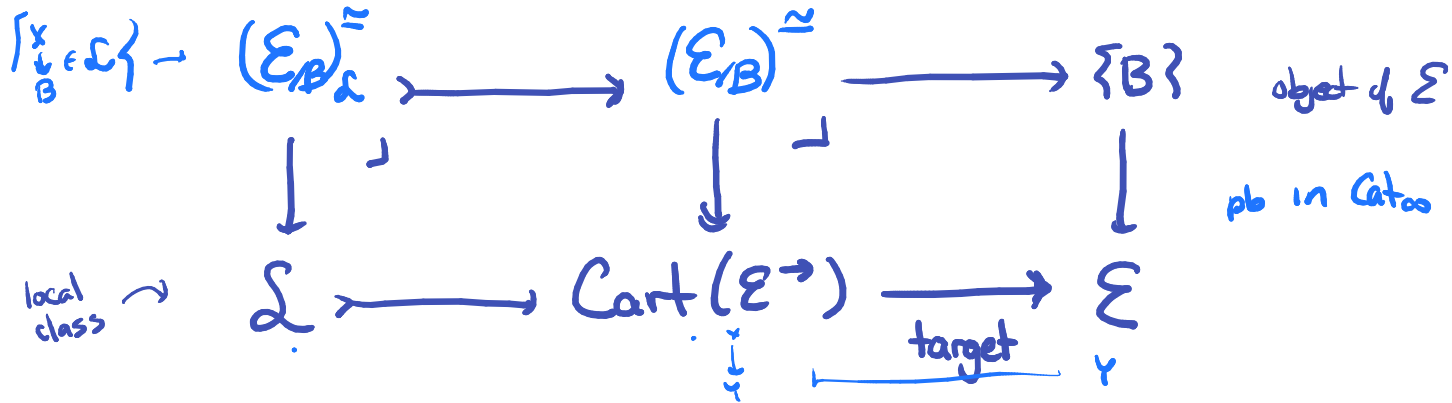
Essential image  $\mathcal{L} := \mathcal{L}_u \subseteq \text{Cart}(\mathcal{E} \rightarrow \Sigma)$  is a local class:

(1)  $f \rightarrow f' \in \text{Cart}(\mathcal{E} \rightarrow \Sigma)$ ,  $f' \in \mathcal{L} \implies f \in \mathcal{L}$  closed under base change

(2)  $\mathcal{L}$  has colimits and  $\mathcal{L} \rightarrow \Sigma$  preserves colimits

descent  $\iff \text{Cart}(\mathcal{E} \rightarrow \Sigma)$  is a local class

$u: U_* \rightarrow U$  universal family  $\Rightarrow \underline{\mathcal{L}} = \underline{\mathcal{L}}_u$  is a bounded local class:



bounded means each  $(\mathcal{E}/B)_{\underline{\mathcal{L}}}^{\cong}$  is essentially small co-groupoid

$$\left[ (\mathcal{E}/B)_{\underline{\mathcal{L}}}^{\cong} \simeq \text{Map}_{\mathcal{E}}(B, U) \right]$$

$\mathcal{E}$  a-topos

Prop: Correspondence

$$\left\{ \begin{array}{l} \text{universal families} \\ u: U_{\star} \rightarrow U \text{ in } \text{Cart}(\mathcal{E}^{\rightarrow}) \end{array} \right\} \begin{array}{c} \xrightarrow{\quad} \\ \xLeftrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \left\{ \begin{array}{l} \text{banded local classes} \\ \mathcal{L} \subseteq \text{Cart}(\mathcal{E}^{\rightarrow}) \end{array} \right\}$$

Idea:  $B \mapsto (\mathcal{E}/B)_{\mathcal{L}}^{\wedge}$  is a functor  $\mathcal{E}^{\text{op}} \rightarrow \mathcal{S}$   
 $\omega$ -gpts

univ colim + descent  $\Rightarrow$  preserves limits,  $\mathcal{E}$  presentable  $\Rightarrow$  representable by  $U$

Prop: Every morphism in an  $\omega$ -topos is contained  
in some bounded local class

Proof idea:  $\text{Cart}(\mathcal{E}^{\rightarrow}) = \bigcup \mathcal{L}^{\kappa}$

$\mathcal{L}^{\kappa}$  = bounded local class of "relatively  $\kappa$ -compact"  
morphisms

↑  
regular  
cardinal

Lurie, Gepner-Koch, Rasekh

→ get an exhaustive collection  $\{ U_*^k \rightarrow U^k \}$   
of universal families  
→ the "union" (in a "higher universe"):  $\Omega_* \rightarrow \Omega$   
universal map

Theorem:  $\mathcal{E}$  is an  $\infty$ -topos iff  
presentable, univ. colimits, "enough" universal families

Univalence :

$p: E \rightarrow B$  in  $\mathcal{E}$   $\implies$   $\exists$   $\omega$ -topos (or 1-topos)

$$B \xrightarrow{\text{id.}!} \text{Iso}(p) \xrightarrow{\pi} B \times B$$

$$\left\{ \begin{array}{ccc} T & \overset{\dots}{\dashrightarrow} & \text{Iso}(p) \\ & \xrightarrow{(f, g)} & \downarrow \pi \\ & & B \times B \end{array} \right\}$$

$\iff$

$$\left\{ \begin{array}{ccc} f^*E & \overset{\text{iso}}{\dashrightarrow} & g^*E \\ & \searrow & \swarrow \\ & T & \end{array} \right\}$$

$p: E \rightarrow B$  is univalent iff  $B \xrightarrow{\pi} \text{Iso}(p)$  is iso

Thm:  $p$  univ family  $\iff p$  is univalent.

Van Kampen colimits:  $\mathcal{E}$   $\infty$ -category with pullbacks

$$f: A \rightarrow B \text{ in } \mathcal{E} \quad \rightsquigarrow \quad f^*: \mathcal{E}/_B \rightarrow \mathcal{E}/_A$$

A colimit cone  $F: J * \Delta^0 \rightarrow \mathcal{E}$  is a van Kampen colimit

if induced functor

$$(J * \Delta^0)^{op} \longrightarrow \widehat{\text{Cat}}_{\infty}, \quad j \longmapsto \mathcal{E}/_{F(j)}$$

is a limit cone.



Example: In a 1-topos,  $\mathcal{E}$  coproducts are van Kampen

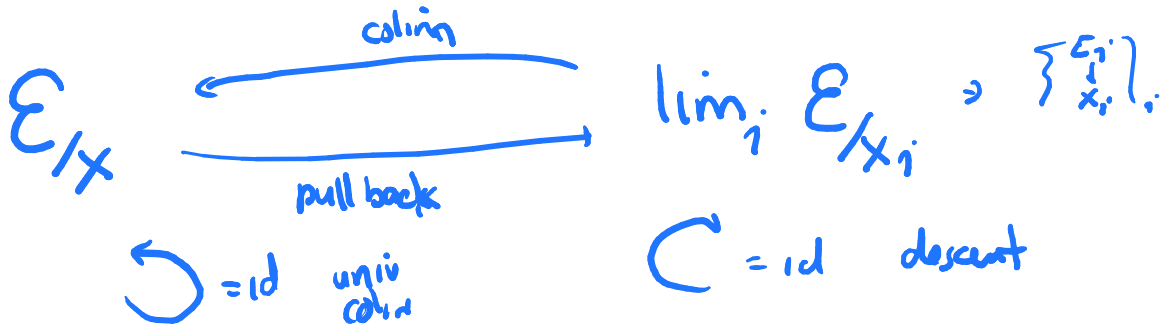
i.e., if  $X = \coprod X_i$ ,  $t \in \mathcal{E}$

$$\mathcal{E}_{/X} \xrightarrow{\cong} \prod_i \mathcal{E}_{/X_i} \quad \checkmark$$

in General, pushouts are not van Kampen in a 1-topos

Theorem:  $\Sigma$  is an  $\infty$ -topos iff  
 presentable, all colimits are van Kampen

Proof idea: if  $X \simeq \operatorname{colim}_{i \in I} X_i \in \mathcal{E}$



Have  $\mathcal{E}^{\text{op}} \rightarrow \widehat{\text{Cat}}_{\infty}$ ,  $A \mapsto \mathcal{E}/A$

For each bounded local class  $\mathcal{L}$ , have

$\mathcal{E}^{\text{op}} \rightarrow \text{Cat}_{\infty}$ ,  $A \mapsto (\mathcal{E}/A)_{\mathcal{L}} \subseteq \mathcal{E}/A$   
← small  $\infty$ -category

functor preserves limits,  $\mathcal{E}$  presentable

$\Rightarrow$  representable by an internal  $\infty$ -category object

$U^{\mathcal{L}}$  of  $\mathcal{E}$

Theorem:  $\mathcal{E}$  is an  $\infty$ -topos iff  
presentable, every morphism in a local class  
represented by internal  $\infty$ -category object

Rasekh, "A theory of elementary higher toposes" (arXiv 2018)

Elementary  $\infty$ -topoi ?

What <sup>does</sup> "elementary" mean?

Definitions by Shulman, Rasekh.

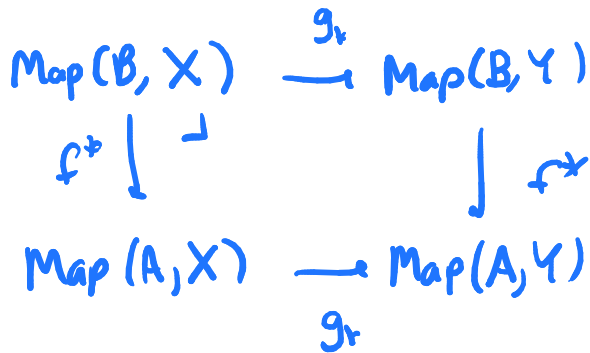
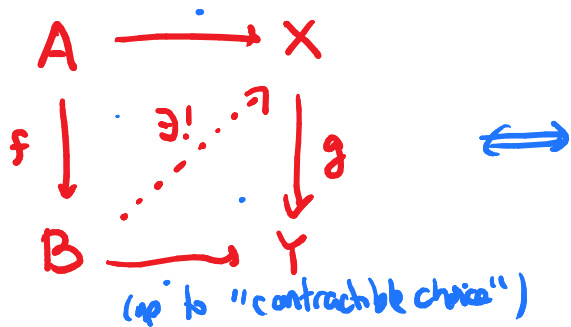
- finite complete and cocomplete
- has subobject classifier
- every morphism in a local class represented by internal  $\infty$ -category object.

$\Rightarrow$  loc Cart closed

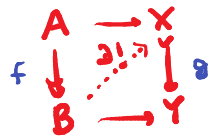
Monomorphism:  $f: A \rightarrow B$  such that  $\Delta(f): A \rightarrow A \times_B A$  iso

Orthogonal maps:  $f \perp g$  if

unique lift exists in any



Cover:  $f: A \rightarrow B$  such that  $f \perp g \quad \forall g \in \underline{\text{Mono}}$



(or "effective epimorphism")

(or "surjection")

usually not epimorphisms!

Example: In  $\mathcal{S} = \infty\text{-groupoids}$ , (= htop of spaces)

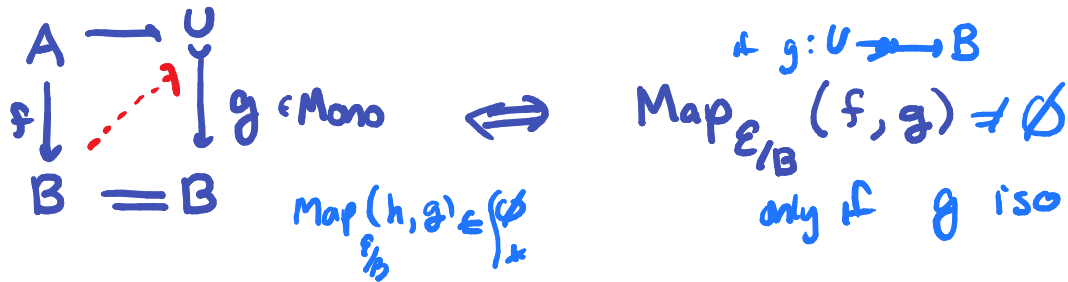
$f: A \rightarrow B$  cover iff  $\pi_0 A \rightarrow \pi_0 B$  surjective

sets of path components

"lift points up to homotopy"

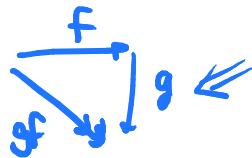
Observation: Mono is stable under pullback. so

$f: A \rightarrow B$  cover iff ~~unique~~ lift exists in every



$\Rightarrow$   $f$  cover in  $\mathcal{E}$  iff cover in  $\mathcal{E}/B$

$\Rightarrow$   $gf$  cover  $\Rightarrow$   $g$  cover.





Consequence: If  $F: I \rightarrow \mathcal{E}$  has colimit  $B$ ,

then  $\coprod_{i \in I_0} F(i) \rightarrow B$  is a cover.

Proof: WLOG  $B=1$ . Let  $U \rightarrow 1$ . Then

$$\begin{array}{ccc} \coprod F(i) \rightarrow U & & \\ \downarrow \quad \text{?} \quad \downarrow & \Rightarrow & \\ 1 = 1 & & \end{array}$$

$$\begin{aligned} \text{Map}(1, U) &\simeq \text{Map}(\text{colim}_i F(i), U) \\ &\simeq \lim_i \text{Map}(F(i), U) \end{aligned}$$

In a presentable  $\infty$ -category,  $(\text{Cover}, \text{Mono})$  is a factorization system:

- $\text{Cover} \perp \text{Mono}$

- $\text{Cover}, \text{Mono}$  stable under retracts.

- Every  $f$  admits  $f = i \circ p$ .  $p \in \text{Cover}$   
 $i \in \text{Mono}$

$f: A \rightarrow B$

essentially  
unique:

$$A \xrightarrow{p} \text{Im}(f) \xrightarrow{i} B$$

In an  $\infty$ -topos, construct cover/mono factorization

Čech nerve:  $f: A \rightarrow B \rightsquigarrow C_f: \Delta_+^{\infty} \rightarrow \mathcal{E}$

$\Delta^{\text{op}} \rightarrow \Delta^0$  aug. simplicial object

$$C_f: \dots \rightrightarrows A \times_B A \times_B A \rightrightarrows A \times_B A \rightrightarrows A \xrightarrow{f} B$$

$$\Rightarrow A \xrightarrow{p} \operatorname{colim}_{\Delta^{\text{op}}} C_f \xrightarrow{i} B$$

$\operatorname{Im}(f)$

Proof: WLOG,  $B=1$  ( $E \rightsquigarrow E/B$ )  $f: A \rightarrow 1$

$C_f$ :  $\dots \cong A \times A \times A \cong A \times A \cong A \rightarrow 1$  ↓

$A \times C_f$ :  $\dots \cong A \times A \times A \times A \cong A \times A \times A \cong A \times A \rightarrow A$  ↓  
"contracting homotopy" ( $\Delta_+^{op} \xrightarrow{\varepsilon} \Delta_0^{op}$ ) .

"absolute colimit" every composite  $\Delta_+^{op} \rightarrow \Delta_0^{op} \rightarrow \mathcal{C}$   
is a colimit cone in  $\mathcal{C}$ .

so (1)  $A \times E = \operatorname{colim}_{\Delta_{\text{op}}} A \times C_f \xrightarrow{\sim} A$   $E = \operatorname{colim}_{\operatorname{colim}_{A^{\times n}} C_f}$

$\uparrow$  univ. colim

$\Rightarrow$  (2)  $E \times E = \operatorname{colim}_{\Delta_{\text{op}}} C_f \times E \cong \operatorname{colim}_{\Delta_{\text{op}}} C_f = E$  ✓

so  $E \xrightarrow{\sim} E \times E \Rightarrow \underline{E \xrightarrow{\sim} 1}$

(3)  $E \cong \operatorname{colim}_{\Delta_{\text{op}}} C_f \Rightarrow \coprod_{n \geq 0} A^{\times n+1} \xrightarrow{\text{conv.}} E$

$\swarrow$   $\searrow$   $\nearrow$   $\searrow$

$A$   $P$   $\Rightarrow P \text{ conv.}$

Covers and local classes:  $\mathcal{L}$  is a local class iff  $\stackrel{\text{Cont}(\mathcal{E} \rightarrow)}{=}$

(1)  $\mathcal{L}$  closed under coproducts

and  $\forall$  
$$\begin{array}{ccc} X & \longrightarrow & Y \\ f \downarrow \sim & & \downarrow g \\ A & \xrightarrow{p} & B \end{array}$$
 pullback.

(2)  $g \in \mathcal{L} \implies f \in \mathcal{L}$

(3) if  $p \in \text{Cover}$ ,  $f \in \mathcal{L} \implies g \in \mathcal{L}$

Consequence: given  $f: E \rightarrow B$ ,  $\in \mathcal{E}$ , let

$$\mathcal{L}^f := \left\{ g: E'' \rightarrow B'' \text{ st } \exists \begin{array}{ccccc} E & \longleftarrow & E' & \longrightarrow & E'' \\ & \swarrow & \downarrow & \searrow & \downarrow g \\ B & \longleftarrow & B' & \xrightarrow{\text{cover}} & B'' \end{array} \right\}$$

"g looks locally like f"

$\Rightarrow$   $\mathcal{L}$  is the smallest local class containing  $f$ , (banded)

$\Rightarrow \exists U_*^f \rightarrow U^f$  univ. family

# Proof of cover characterization of local class:

$(\Rightarrow)$ : local class  $\Rightarrow$  (1)  $\perp$  closed, (2) base change

• balen cover

$g \in \mathcal{L}, f \in \mathcal{C}$

(3) pull back along cover:

$$\Rightarrow X \times_Y X \times_Y X \Rightarrow X \times_Y X \Rightarrow X \xrightarrow{f} Y \xleftarrow{\text{colim (univ colim)}} \left( \begin{array}{c} \downarrow f \\ \downarrow g \end{array} \right) \Rightarrow f \in \mathcal{C} \Rightarrow g \in \mathcal{L}$$

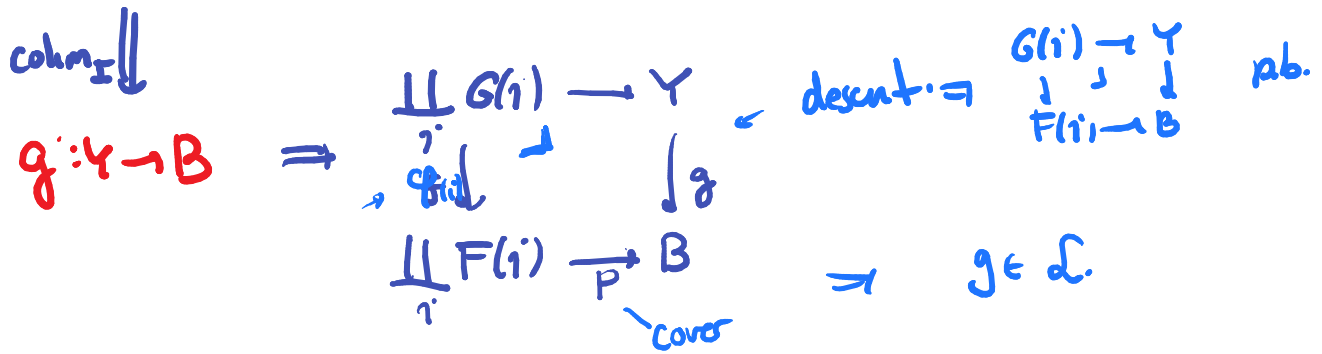
$$\Rightarrow A \times_B A \times_B A \Rightarrow A \times_B A \Rightarrow A \xrightarrow{p} B \xleftarrow{\text{colim}} \left( \begin{array}{c} \uparrow \text{tech new} \\ \uparrow \text{cover} \end{array} \right)$$



Proof (a'd)  $\Leftrightarrow$ : (1), (2), (3)  $\stackrel{?}{\Rightarrow}$   $\mathcal{L}$  closed under colimits in  $\text{Cat}(\mathcal{E} \rightarrow)$

$\downarrow$   $\uparrow$   
Coproducts

$\gamma: G \rightarrow F$  Cartesian nat transf of  $I \rightarrow \mathcal{E}$   
 each  $G(i) \xrightarrow{\rho(i)} F(i) \in \mathcal{L}$  ( $I \rightarrow \mathcal{L} \in \text{Cat}(\mathcal{E} \rightarrow)$ )



These ideas also show:

Mono and Cover are local classes in  $\text{Cart}(\mathcal{E} \rightrightarrows)$

Proof:

(1)  $\perp$  stable :

Mono

descent

Cover

$\perp$  to Mono

(2) base change :

easy

descent + Čech nerve

(3) p.b along Cover :

descent +  
Čech nerve

closed under composition,  
 $gf \in \text{Cover} \Rightarrow g \in \text{Cover}$ .

Truncation: In an  $\infty$ -category  $\mathcal{C}$

•  $f: A \rightarrow B$  is  $n$ -truncated if  $\Delta^{n+2}(f): A \rightarrow \Omega^{n+2}(f)$   
is iso

•  $X$   $n$ -truncated object  $\Leftrightarrow X \rightarrow 1$  is  $n$ -truncated.

•  $\mathcal{C}_{\leq n} \subseteq \mathcal{C}$  full subcategory of  $n$ -truncated objects

$B \in \mathcal{C}_{\leq n} \Rightarrow \text{Map}_{\mathcal{C}}(A, B) \in \mathcal{S}_{\leq n}$  so  $\mathcal{C}_{\leq n}$  is an " $(n+1)$ -category"  
"n-groupoid" " $(n+1, 1)$ -category"

$$\text{Iso} \subseteq \text{Mono} \subseteq \text{Trunc}_0 \subseteq \text{Trunc}_1 \subseteq \text{Trunc}_2 \subseteq \dots \subseteq \mathcal{C}^{\rightarrow}$$

$$\downarrow \perp$$

$$\mathcal{C}^{\rightarrow} \supseteq \text{Cover} \supseteq \text{Conn}_1 \supseteq \text{Conn}_2 \supseteq \text{Conn}_3 \supseteq \dots \supseteq \text{Iso}$$

where  $f: A \rightarrow B$  is  $n$ -connective if  $f \perp \text{Trunc}_{n-1}$   
 (also " $n$ -connected", also " $(n-1)$ -connected")

object  $X$  is  $n$ -connective if  $(X \rightarrow 1) \in \text{Conn}_n$

Properties: (in presentable  $\infty$ -category)

- $(\text{Conn}_{n+1}, \text{Trunc}_n)$  is a factorization system

$$A \xrightarrow{f} B \Rightarrow A \xrightarrow{(n+1)\text{-conn}} \text{Im}_n(f) \xrightarrow{n\text{-trunc}} B \quad \text{relative } n\text{-truncation}$$

$$X \Rightarrow X \xrightarrow{(n+1)\text{-conn}} \text{Tr}_n(X) \xrightarrow{n\text{-trunc}} 1 \quad \text{absolute } n\text{-truncation}$$

- $\text{Conn}_n, \text{Trunc}_n$  are local classes

$$\Rightarrow X \rightarrow \dots \rightarrow \text{Tr}_{n+1}(X) \rightarrow \text{Tr}_n(X) \rightarrow \text{Tr}_{n-1}(X)$$

Example: In  $\mathcal{S}$ :

- $X$   $n$ -truncated iff  $\pi_k(X, x) = 0 \quad \forall x \in X, k > n$
  - $X$   $n$ -connective iff  $\pi_k(X, x) = 0 \quad \forall x \in X, k \leq n$   
( $n \geq 0$ ) and  $X \neq \emptyset$  "  $(n-1)$ -connected space"
  - $f: X \rightarrow Y$  is  $\left\{ \begin{array}{l} n\text{-truncated} \\ n\text{-connective} \end{array} \right.$   $\longrightarrow$  "  $n$ -connected map"
- iff all its hprop fibers are  $\left\{ \begin{array}{l} n\text{-trunc} \\ n\text{-connective} \end{array} \right.$

In  $\mathcal{S}$ ,  $X \rightarrow \mathrm{Tr}_n X$  is constructed by

"killing homotopy in  $\dim > n$ " by

"attaching cells in  $\dim \geq n+2$ "

In  $\mathrm{PSh}(C)$ ,  $\mathrm{Tr}_n X$  computed "pointwise"

In  $\mathcal{E} \begin{array}{c} \xleftarrow{\ell} \\ \xrightarrow{i} \end{array} \mathrm{PSh}(C)$ ,  $\mathrm{Tr}_n^{\mathcal{E}}(X) \cong \ell(\mathrm{Tr}_n^{\mathrm{PSh}(C)}(iX))$

$\ell, i$  preserve "n-truncation" property

Example:

$n$ -gerbes :

(= "EM  $n$ -gerbes")

$$\text{Gerbes}_n := \text{Trunc}_n \cap \text{Conn}_n.$$

$$(\text{Trunc}_n \cap \text{Conn}_{n+1} = \text{Iso})$$

$\mathcal{E}_{\text{Gerbes}_n} \subseteq \mathcal{E}$  : full subcategory spanned by  
 $(E \xrightarrow{p} 1) \in \text{Gerbes}_n$   
 $\uparrow$   
topos

$$(\mathcal{E}_{\text{Gerbes}_n})_* = \{ 1 \xrightarrow{s} E \xrightarrow{p} 1, p \in \text{Gerbes}_n \} \leftarrow \text{is a 1-category.}$$



Prop:  $(\mathcal{E}_{\text{Gerben}})_* \xrightarrow{\sim} (\mathcal{E}_{\leq 0})_{\text{ab}}$   
 ( $n \geq 2$ )

1-category  
 ← abelian group objects  
 in 1-categories  $\mathcal{E}_{\leq 0}$   
 (n=1, group objects, (1-topoi)  
 n=0, pointed objects)

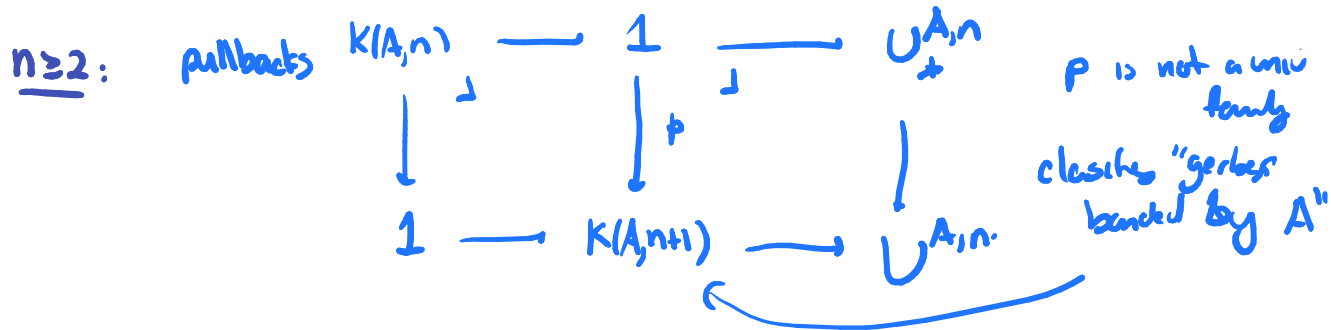
Construction of  $(\mathcal{E}_{\text{Gerben}})_* \xrightarrow{\Omega^n} \mathcal{E}_{\leq 0}$ :

$(1 \xrightarrow{\cdot} E \xrightarrow{\cdot} 1) \Rightarrow \Delta^n(s) : 1 \rightarrow \Omega^n(s), \Omega^n(s) \in \mathcal{E}_{\leq 0}$

Inverse :  $A \mapsto K(A, n) \in (\mathcal{E}_{\text{Gerben}})_*$

Example: Given  $A \in (\mathcal{E}_{\leq 0})_{\text{ob}} \Rightarrow (K(A, n) \xrightarrow{p} 1) \in \mathcal{E}$

$\leadsto$  universal family  $U_*^{A, n} \rightarrow U^{A, n}$  of gerbes  
 "locally like  $p$ "



$\text{Conn}_\infty := \bigcap \text{Conn}_n$        $\infty$ -connected maps

$\iff f: A \rightarrow B$  st  $\text{Im}_n(f) \cong B \quad \forall n$

also a local class

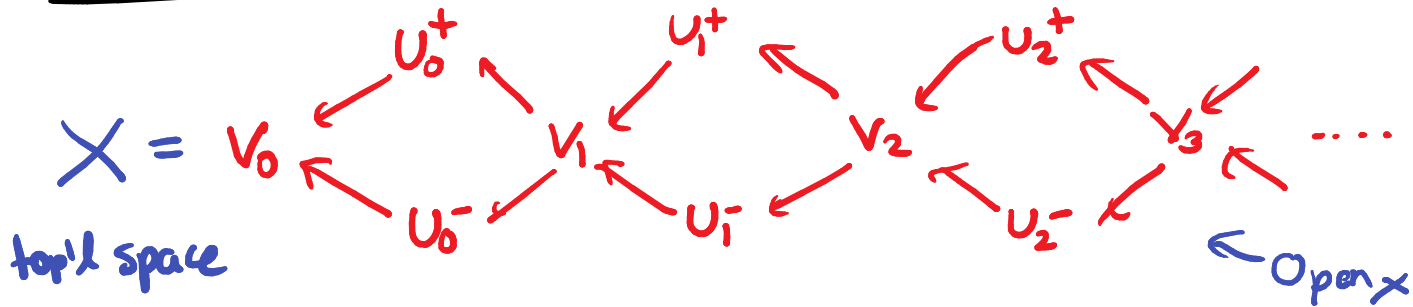
$X$   $\infty$ -connected object if  $(X \rightarrow 1) \in \text{Conn}_\infty$ .

Example: In  $\mathcal{P}, \text{Psh}(C)$ ,  $\text{Conn}_\infty = \text{Iso}$

$\uparrow \nearrow$

"Whitehead theorem"

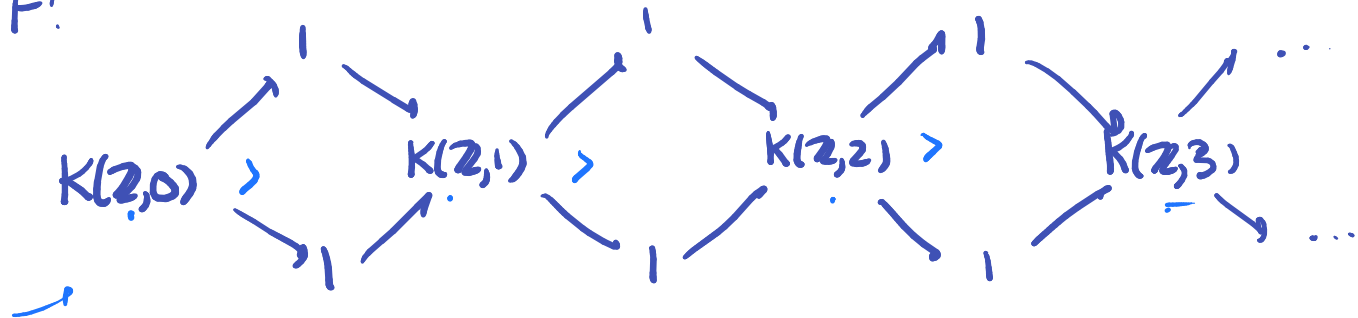
Example: Non-trivial  $\infty$ -connected object in  $\infty$ -topos



$$F: \text{Open}_X^{\text{op}} \rightarrow \mathcal{S} \quad \text{so that} \quad \left\{ \begin{array}{l} F(U_n^\pm) \cong * \\ F(V_n) \cong K(\mathbb{Z}, n) \end{array} \right.$$

Eilenberg-Mac Lane  
Space

F:



so  $F \in \text{Sh}(X) \xrightarrow{\cong} \text{Fun}(\text{Open}_X^{\text{op}}, \mathcal{S})$

$$K(2,n) = \Omega K(2,n+1)$$

$\infty$ -topos  $\mathcal{T}$

$F$  is  $\infty$ -connected,  $F \neq *$

$$\left. \begin{array}{l} \text{Tr}_m^{\text{sh}} F = \ell(\text{Tr}_m^{\text{Psh}} F) \\ \text{Tr}_m(K(2,n)) = * \quad \forall n > m \end{array} \right\}$$

Next time :

- Grothendieck topologies ?
- Geometric morphisms