The over-topos at a model

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Introduction

Introduction

In topology one may look at the *up-set* $\uparrow_{\sqsubseteq} \{x\}$ and *down-set* $\downarrow_{\sqsubseteq} \{x\}$ at a point for the specialization order

$\uparrow_{\sqsubseteq} \{x\}$ contains all points above x - which are contained in any neighborhood of x
$\downarrow_{\sqsubseteq} \{x\}$ contains all points below x - whose neighborhoods contain x

This generalizes also to up-sets and down-set for arbitrary subsets.

In this talk we are interested in the topos-theoretic analog of the down-sets.

Points of a topos

Recall that Grothendieck topoi have *categories* of points

$$\mathrm{pt}(\mathcal{E}) \simeq \mathrm{Geom}[\mathrm{Set}, \mathcal{E}]$$

For $\mathcal{E}$ a Grothendieck topos and $p : \mathrm{Set} \to \mathcal{E}$ we can look at the corresponding *under-category* $p \downarrow \mathrm{pt}(\mathcal{E})$ and *over-category* $\mathrm{pt}(\mathcal{E}) \downarrow p$

Respective analogues of the up-set and down-set:

$$\uparrow_{\sqsubseteq} \{x\} \rightsquigarrow p \downarrow \mathrm{pt}(\mathcal{E}) \qquad \downarrow_{\sqsubseteq} \{x\} \rightsquigarrow \mathrm{pt}(\mathcal{E}) \downarrow p$$

This also generalizes to arbitrary geometric morphisms $f : \mathcal{F} \to \mathcal{E}$:

$$f \downarrow \mathrm{Geom}[\mathcal{F}, \mathcal{E}] \qquad \mathrm{Geom}[\mathcal{F}, \mathcal{E}] \downarrow f$$

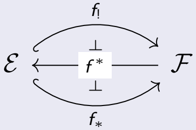
We are interested in the topos whose category of points is the over-category at a given point

Totally connected topos

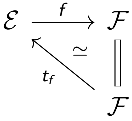
Totally connected geometric morphism

Definition

A geometric morphism f is said to be *totally connected* if f^* has a cartesian left adjoint $f_!$



Equivalently: f has a terminal section given by $t_f = (f_! \dashv f^*)$



Dual to the notion of *local geometric morphism*

Totally connected topos

Definition

In particular a Grothendieck topos $\mathcal{E}$ is *totally connected* if its terminal geometric morphism $!_{\mathcal{E}} : \mathcal{E} \to \mathrm{Set}$ is totally connected.

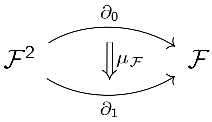
Then $\mathcal{E}$ has a terminal point

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{!_{\mathcal{E}}} & \mathrm{Set} \\ & {}^{t_{\mathcal{E}}}\nwarrow \;\; \simeq & \| \\ & & \mathrm{Set} \end{array}$$

In particular if $\mathcal{E} = \mathrm{Set}[\mathbb{T}]$, $\mathbb{T}[\mathrm{Set}]$ has a terminal object.

Power object and universal codomain

Recall that the 2-category of Grothendieck topoi has *power with 2*:



with ∂_0, ∂_1 the *universal domain* and *universal codomain*

In particular the universal codomain is totally connected

Classifies natural transformations between geometric morphisms into $\mathcal{F}$

$$\mathrm{Geom}[\mathcal{E}, \mathcal{F}]^2 \simeq \mathrm{Geom}[\mathcal{E}, \mathcal{F}^2]$$

In particular its points are morphisms between points of $\mathcal{F}$:

$$\mathrm{pt}(\mathcal{F})^2 \simeq \mathrm{pt}(\mathcal{F}^2)$$

Over-topos

Totally connected geometric morphisms are stable under 2-pullbacks

At any point $p : \mathrm{Set} \to \mathcal{F}$, we can look at the *over-topos* of $\mathcal{F}$ over p classifying morphisms of points over p.

It can be constructed as the pullback of the *universal codomain*

$$\begin{array}{ccc} \mathrm{Set}[p] & \longrightarrow & \mathcal{F}^2 \\ {\scriptstyle u_p}\downarrow & \lrcorner & \downarrow{\scriptstyle \partial_1} \\ \mathrm{Set} & \xrightarrow[p]{} & \mathcal{F} \end{array}$$

We have indeed

$$\mathrm{pt}(\mathrm{Set}[p]) \simeq \mathrm{pt}(\mathcal{F}) \downarrow p$$

We force p to become terminal amongst points of $\mathcal{F}$
The topos $\mathrm{Set}[p]$ is totally connected.

Over-topos (general case)

More generally one can consider the over-topos at a geometric morphism $f : \mathcal{E} \to \mathcal{F}$

$$\begin{array}{ccc} \mathcal{E}[p] & \longrightarrow & \mathcal{F}^2 \\ {\scriptstyle u_f}\downarrow & \lrcorner & \downarrow{\scriptstyle \partial_1} \\ \mathcal{E} & \xrightarrow[f]{} & \mathcal{F} \end{array}$$

We force f to become terminal amongst geometric morphisms into $\mathcal{F}$ up to inverse image

In particular u_f is totally connected
We can also see it as *the totally connected component of* $\mathcal{E}$ *at* f

The universal property

In particular we are interested in the logical aspects of this construction.

If $\mathcal{F}$ is the classifier $\mathrm{Set}[\mathbb{T}]$ of a geometric theory $\mathbb{T}$, the universal property of $\mathcal{E}[M]$

$$\mathbf{Geom}_{\mathcal{E}}[g, u_M] \simeq \mathbb{T}[\mathcal{G}]/g^*(M)$$

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{f} & \mathcal{E}[M] \\ & \searrow^{g} \quad \simeq \quad \swarrow_{u_M} & \\ & \mathcal{E} & \end{array} \qquad \mapsto \qquad N \xrightarrow{f} g^*M$$

One forces M to become terminal in the category of $\mathbb{T}$-models in $\mathcal{E}$.

A site description ?

The purpose of this work is to provide a site description of the over-topos.

The over-topos is a universal construction done at the invariant level of Grothendieck topos: free of any site presentation.

In the case of the classifying topos $\mathrm{Set}[\mathbb{T}]$ of a geometric theory $\mathbb{T}$ and a model $M : \mathcal{E} \to \mathrm{Set}[\mathbb{T}]$, it does not retain any model theoretic information about M nor $\mathbb{T}$.

We would like to construct a site presentation of the over-topos $\mathcal{E}[M]$ expressing the logical aspects of this construction.

We shall split our work in two steps:

- First we construct a site for the over-topos at a set-valued model
- Constructing the over-topos in the general case requires additional technology about *stacks*

Over-topos of a set-valued model

Syntactic site of a geometric theory

(Geometric) syntactic site of $\mathbb{T}$: $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ with

- objects: $\{\vec{x}^{\vec{A}}.\phi\}$ with $\phi(\vec{x}^{\vec{A}})$ a geometric formula with free variables $\vec{x}^{\vec{A}}$ of sort $\vec{A}$
- morphisms: $\mathbb{T}$-provability equivalence classes of $\mathbb{T}$-provably functional formulas

$$\{\vec{x}_1^{\vec{A}_1}.\phi_1\} \xrightarrow{[\theta]} \{\vec{x}_2^{\vec{A}_2}.\phi_2\}$$

- $J_{\mathbb{T}}$ the syntactic topology generated by the basis $\mathcal{B}_{\mathbb{T}}$ of families

$$\left(\ \{\vec{x}_i^{\vec{A}_i}.\phi_i\} \xrightarrow{[\theta_i]} \{\vec{x}^{\vec{A}}.\phi\}\ \right)_{i\in I}$$

coding for all $\mathbb{T}$-provable geometric sequents

$$\phi \vdash_{\vec{x}} \bigvee_{i\in I} (\exists \vec{x_i})\theta_i(\vec{x_i}, \vec{x})$$

Category of elements of a set-valued model

A model M in $\mathbb{T}[\mathrm{Set}]$ is a $J_{\mathbb{T}}$-flat functor $M : \mathcal{C}_{\mathbb{T}} \to \mathrm{Set}$ sending a formula $\{\vec{x}^{\vec{A}}.\phi\}$ on its *interpretation* $[\![\vec{x}^{\vec{A}}.\phi]\!]_M$ in M

Definition (Category of elements)

The *category of elements* of M is the category $\int M$ with

- objects: $(\{\vec{x}^{\vec{A}}.\phi\}, \vec{a})$ with $a : 1 \to [\![\vec{x}^{\vec{A}}.\phi]\!]_M$ a global element
- morphisms: $(\{\vec{x}_1^{\vec{A}_1}.\phi_1\}, \vec{a_1}) \to (\{\vec{x}_2^{\vec{A}_2}.\phi_2\}, \vec{a_2})$ such that $[\![\theta]\!]_M(\vec{a_1}) = \vec{a_2}$, that is

$$\begin{array}{ccc} & 1 & \\ {}^{\vec{a_1}}\swarrow & & \searrow^{\vec{a_2}} \\ [\![\vec{x}_1^{A_1}.\phi_1]\!]_M & \xrightarrow{[\![\theta]\!]_M} & [\![\vec{x}_2^{A_2}.\phi_2]\!]_M \end{array}$$

Equipped with a discrete opfibration $\int M \to \mathcal{C}_{\mathbb{T}}$

Global elements for set-valued models

1 is a generator in Set: hence for M in $\mathbb{T}[\mathrm{Set}]$

$$[\![\vec{x}^{\vec{A}}.\phi]\!]_M = \coprod_{\vec{a}:1\to[\![\vec{x}^{\vec{A}}.\phi]\!]_M} 1$$

By preservation of coproducts along inverse images, we have for any $\mathcal{G}$

$$[\![\vec{x}^{\vec{A}}.\phi]\!]_{\gamma^* M} = \gamma^*([\![\vec{x}^{\vec{A}}.\phi]\!]_M) = \coprod_{\vec{a}:1\to[\![\vec{x}^{\vec{A}}.\phi]\!]_M} 1$$

Global elements determine the interpretations, even along inverse images.

Antecedent topology

As M is $J_{\mathbb{T}}$-flat, we have for each $J_{\mathbb{T}}$-cover an epi

$$\coprod_{i \in I} [\![\vec{x}_i^{\vec{A}_i}.\phi_i]\!]_M \xrightarrow{\langle [\![\theta_i]\!]_M \rangle_{i \in I}} [\![\vec{x}^{\vec{A}}.\phi]\!]_M$$

Definition (Antecedents Topology)

$\int M$ can be equipped with a pretopology $\mathcal{B}_M^{Ant}$:

$$\left((\vec{b}, \{\vec{x}_i^{\vec{A}_i}.\phi_i\}) \xrightarrow{[\theta_i]} (\vec{a}, \{\vec{x}^{\vec{A}}.\phi\}) \right)_{i \in I,\, \vec{b} \,\mid\, [\![\theta_i]\!]_M(\vec{b}) = \vec{a}}$$

with $([\theta_i])_{i \in I}$ a $J_{\mathbb{T}}$-cover.

The *antecedents topology* J_M^{Ant} is the topology generated by $\mathcal{B}_M^{Ant}$

Indexed by the fibers $\coprod_{i \in I} [\![\theta_i]\!]_M^{-1}(\vec{a})$
$\rightarrow$ Gather all antecedents of a along interpretations of $J_{\mathbb{T}}$-covers

A site for the over-topos

Theorem (O.C. & A.O.)

For a $\mathbb{T}$-model M in Set *we have a site presentation of the over-topos*

$$\mathsf{Set}[M] \simeq \mathsf{Sh}(\int M, J_M^{Ant})$$

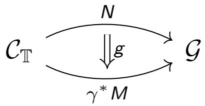
Recall the universal property of the over-topos: for $\mathcal{G}$ with global section functor $\gamma : \mathcal{G} \to \mathsf{Set}$

$$\mathbf{Geom}[\mathcal{G}, \mathsf{Set}[M]] \simeq \mathbb{T}[\mathcal{G}]/\gamma^* M$$

We must prove that any geometric morphism $f : \mathcal{G} \to \mathsf{Sh}(\int M, J_M^{Ant})$ is the name of a homomorphism $g : N \to \gamma^* M$ of $\mathbb{T}$-models in $\mathcal{G}$

From homomorphism to geometric morphism

Suppose we have a homomorphism $g : N \to \gamma^* M$ in $\mathbb{T}[\mathcal{G}]$
$\to$ Same thing as a natural transformation between $J_{\mathbb{T}}$-flat functors



with components

$$[\![\vec{x}^{\vec{A}}.\phi]\!]_N \xrightarrow{g_{\{\vec{x}^{\vec{A}}.\phi\}}} [\![\vec{x}^{\vec{A}}.\phi]\!]_{\gamma^*(M)}$$

Those are the data from which we shall compute a J_M^{Ant}-flat functor

$$\int M \xrightarrow{\overline{g}} \mathcal{G}$$

Constructing a flat functor from fibers

At each $(\{\vec{x}^{\vec{A}}.\phi\}, a)$ of $\int M$ one can take the pullback

$$
\begin{array}{ccc}
N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\phi\}} & \longrightarrow & 1 \\
\big\downarrow & \lrcorner & \big\downarrow \gamma^*(\vec{a}) \\
[\![\vec{x}^{\vec{A}}.\phi]\!]_N & \xrightarrow[g_{\{\vec{x}^{\vec{A}}.\phi\}}]{} & [\![\vec{x}^{\vec{A}}.\phi]\!]_{\gamma^*(M)}
\end{array}
$$

while naturality of g yields at each morphism $[\theta]$ a morphism

$$N^{\vec{a_1}}_{\{\vec{x}_1^{\vec{a_1}}.\phi_1\}} \overset{\vec{N}^{\vec{a_1},\vec{a_2}}_{[\theta]}}{\longrightarrow} N^{\vec{a_2}}_{\{\vec{x}_2^{\vec{a_2}}.\phi_2\}}$$

induced by the universal property of pullback

Constructing a flat functor from fibers

From this we define a functor

$$\begin{array}{ccc} \int M & \overset{\overline{g}}{\to} & \mathcal{G} \\ (\vec{a}, \{\vec{x}^{\vec{A}}.\phi\}) & \mapsto & N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\phi\}} \\ [\theta] & \mapsto & N^{\vec{a_1},\vec{a_2}}_{[\theta]} \end{array}$$

Proving J_M^{Ant}-flatness relies mostly on *stability of coproducts and epi*: it will ensure that for each antecedent family one has an epi in $\mathcal{G}$

$$\coprod_{\vec{b} \in \langle [\![\theta_i]\!]_M \rangle^{-1}_{i \in I}(\vec{a})} N^{\vec{b}}_{\{\vec{x}_i^{\vec{A}_i}.\phi_i\}} \to N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\phi\}}$$

Cartesianness comes from the constructions of $\overline{g}$ from pullbacks.

From geometric morphisms to morphisms of models

Conversely, for a geometric morphism $f : \mathcal{G} \to \mathrm{Sh}(\int M, J_M^{Ant})$
$\to$ same thing as a J_M-flat functor $\int M \to \mathcal{G}$ defining for each element of M an object $N^{\vec{a}}_{\{\vec{x}_1^{\vec{A}}.\phi\}}$ and transitions morphisms between them.

Define at each element a canonical map through its inverse image

$$\begin{array}{ccc}
N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\phi\}} & \xrightarrow{\;!\;} & 1 \\
 & \searrow^{g^{\vec{a}}_{\{\vec{x}^{A}.\phi\}}} & \downarrow{\scriptstyle \gamma^*(\vec{a})} \\
 & & [\![\vec{x}^{\vec{A}}.\phi]\!]_{\gamma^*(M)}
\end{array}$$

Then glue the objects corresponding to elements of a same interpretation

$$\begin{array}{ccc}
N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\phi\}} & & \\
\downarrow & \searrow^{g^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\phi\}}} & \\
\coprod_{\vec{a} \in [\![\vec{x}^{A}.\phi]\!]_M} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\phi\}} & \overset{\dashrightarrow}{g_{\{\vec{x}^{\vec{A}}.\phi\}}} & [\![\vec{x}^{\vec{A}}.\phi]\!]_{\gamma^*(M)}
\end{array}$$

From geometric morphisms to morphisms of models

For each formula in context $\{\vec{x}^{\vec{A}}.\phi\}$ define

$$N_{\{\vec{x}^{\vec{A}}.\phi\}} = \coprod_{a \in [\![x^{\vec{A}}.\phi]\!]_M} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\phi\}}$$

This provides us a functor

$$\begin{array}{ccc} \mathcal{C}_{\mathbb{T}} & \xrightarrow{F_N} & \mathcal{G} \\ \{\vec{x}^{\vec{A}}.\phi\} & \mapsto & N_{\{\vec{x}^{\vec{A}}.\phi\}} \end{array}$$

together with a natural transformation

$$g = (g_{\{\vec{x}^{\vec{A}}.\phi\}})_{\{\vec{x}^{\vec{A}}.\phi\} \in \mathcal{C}_{\mathbb{T}}}$$

Reversibility of the process

Those assignments are mutual inverse

- For $g : N \to \gamma^* M$ we have a fiber decomposition

$$N_{\{\vec{x}^{\vec{A}}.\phi\}} \simeq \coprod_{\vec{a} \in [\![\vec{x}^A.\phi]\!]_M} N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\phi\}}$$

- For a J_M^{Ant}-continuous $N_{(-)} : \int M \to \mathcal{G}$, one has by stability

$$\gamma^*(\vec{a})^*(\coprod_{\vec{b} \in [\![\vec{x}^A.\phi]\!]_M} N^{\vec{b}}_{\{\vec{x}^{\vec{A}}.\phi\}}) \simeq \coprod_{\vec{b} \in [\![\vec{x}^A.\phi]\!]_M} \gamma^*(\vec{a})^*(N^{\vec{b}}_{\{\vec{x}^{\vec{A}}.\phi\}})$$

  Then disjointness of coproducts in Grothendieck toposes ensures that each pullback

$$\gamma^*(\vec{a})^*(N^{\vec{b}}_{\{\vec{x}^{\vec{A}}.\phi\}})$$

  is not empty only if $a = b$. Hence theses fibers do not merge and we can recover N.

Logical aspect of the over-topos

The over-theory of M

$\mathbf{Set}[M]$ classifies a certain theory $\mathbb{T}_M$ axiomatizing the $\mathbb{T}$-model homomorphisms to (internalizations of) M:

$$\mathbb{T}_M[\mathcal{G}] \simeq \mathbb{T}[\mathcal{G}]/\gamma^* M$$

This requires a new *over-language* Set_M together with an auxiliary language $\mathcal{L}_M$ and an interpretation $\sharp : \mathrm{Set}_M \to \mathcal{L}_M$

Morally, we are going to construct $\mathbb{T}_M$ in the language Set_M from sequents whose interpretations will be tested on a canonical $\mathcal{L}_M$-structure associated to M.

The over-language of $\mathbb{T}_M$ and the auxiliary language

The language of the over theory is constructed from $\int M$:

Definition

Define the over-language Set_M having a sort $S_{(\vec{a},\{\vec{x}^{\vec{A}}.\phi\})}$ for each object $(\vec{a},\{\vec{x}^{\vec{A}}.\phi\})$ in $\int M$, and function symbol

$$S_{(\vec{a_1},\{\vec{x_1}^{\vec{A_1}}.\phi_1\})} \xrightarrow{f_\theta^{\vec{a_1},\vec{a_2}}} S_{(\vec{a_2},\{\vec{x_2}^{\vec{A_2}}.\phi_2\})}$$

for each $[\theta(\vec{x_1}^{\vec{a_1}},\vec{x_2}^{\vec{a_2}})]_{\mathbb{T}} : \{\vec{x_1}^{\vec{a_1}}.\phi_1\} \longrightarrow \{\vec{x_2}^{\vec{a_2}}.\phi_2\}$ such that $[\![\theta]\!]_M(\vec{a_1}) = \vec{a_2}$

Definition

The auxiliary language $\mathcal{L}_M$ is the extension of $\mathcal{L}$ containing a new constant symbol $c_{(\vec{a},\{x^{\vec{A}}.\phi\})}$ of sort $\vec{A}$ for each $(\vec{a},\{x^{\vec{A}}.\phi\}) \in \int M$.

M has a canonical $\mathcal{L}_M$ structure M^c where $c_{(\vec{a},\{x^{\vec{A}}.\phi\})}$ is interpreted by a.

The over-theory

Construct the interpretation $\sharp : \mathsf{Set}_M \to \mathcal{L}_M$ by substituting any free variable of sort $S_{(\vec{a},\{\vec{x}^{\vec{A}}.\phi\})}$ with $c_{(\vec{a},\{x^{\vec{A}}.\phi\})}$

Definition

$\mathbb{T}_M$ is the theory over Set_M having as axioms all the geometric sequents

$$\phi \vdash_{\vec{x}^{\vec{S}_{(\vec{a},\{\vec{x}^{\vec{A}}.\phi\})}}} \psi$$

whose interpretation in $\mathcal{L}_M$

$$\phi^\sharp \vdash \psi^\sharp$$

is valid in M^c.

Somewhat reminiscent of the *complete diagram* of a model.

The over-topos as the classifier of the over-theory

Theorem (O.C.)

The over-topos is geometrically equivalent to the classifier of the over-theory at M:

$$\mathrm{Sh}(\int M, J_M^{\mathrm{ant}}) \simeq \mathrm{Sh}(\mathcal{C}_{\mathbb{T}_M}, J_{\mathbb{T}_M})$$

There is a full and faithful canonical functor

$$\begin{array}{ccc} \int M & \xrightarrow{V} & \mathcal{C}_{\mathbb{T}_M} \\ (\vec{a}, \{\vec{x}^{\vec{A}}.\phi\}) & \mapsto & \{x^{S_{(\vec{a},\{\vec{x}^{\vec{A}}.\phi\})}}.\top\} \\ [\theta] & \mapsto & f_\theta^{\vec{a_1},\vec{a_2}} \end{array}$$

V defines a *dense* morphism of sites $(\int M, J_M^{\mathrm{ant}}) \to (\mathcal{C}_{\mathbb{T}_M}, J_{\mathbb{T}_M})$
Hence the associated sheaf topoi are equivalent.

Over-topos at an arbitrary geometric morphism

Insuffisence of global elements in the general case

We used that $1 = y_*$ is generator in $\mathrm{Set} = \mathrm{Sh}(*)$

But in an arbitrary Grothendieck topos $\mathcal{E}$, 1 is not anymore a generator ! One must consider *generalized elements*.

If $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{C}, J)$ (with J subcanonical), objects of $\mathcal{C}$ form a small generator.

Hence one can at least restrict to *basic* generalized elements of the form $a : y_c \to E$.

In particular, for a model M in an arbitrary Grothendieck topos $\mathcal{E}$, that is $f_M : \mathcal{C}_{\mathbb{T}} \to \mathcal{E}$, one must consider generalized elements of interpretations

$$E \xrightarrow{a} [\![x^{\vec{A}}.\phi]\!]_M$$

Why do we need stacks ?

We must allow both the indexing object of generalized elements and the formula in context in an interpretation to vary

we shall have to consider jointly the comma categories $E \downarrow f_M$ and $\mathcal{E} \downarrow [\![x^{\vec{A}}.\phi]\!]_M$.

A model actually defines a stack over the site of the base topos

Giraud topology

Let be $\mathbb{M} : \mathcal{E}^{\mathrm{op}} \to$ Cat a cartesian stack on a Grothendieck topos (for the canonical topology)

Then $\int \mathbb{M}$ can be equipped with the *Giraud topology* $J_{\mathbb{M}}^{Gir}$, which is the smallest topology making the fibration of $\mathbb{M}$ a comorphism of site

$$(\int \mathbb{M}, J_{\mathbb{M}}^{Gir}) \xrightarrow{\pi_{\mathbb{M}}} (\mathcal{E}, J_{Can})$$

Giraud topology

If $(\mathcal{C}, J)$ is a small standard site for $\mathcal{E}$, then restricting $\mathbb{M}$ along the Yoneda embedding provides a cartesian stack $\mathbb{M}y : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ for J. On can restrict $J_{\mathbb{M}}^{Gir}$ on $\int \mathbb{M}y \to \mathcal{C}$ to get a comorphism of site

$$(\int \mathbb{M}y, J_{\mathbb{M}y}^{Gir}) \xrightarrow{\pi_{\mathbb{M}y}} (\mathcal{C}, J)$$

$J_{\mathbb{M}y}^{Gir}$ is generated from families of cartesian lifts of J-covers.

Then one has a geometric morphism

$$\mathrm{Sh}(\int \mathbb{M}, J_{\mathbb{M}}^{Gir}) \xrightarrow{p_{\mathbb{M}}} \mathcal{E}$$

and a geometric equivalence

$$\mathrm{Sh}(\int \mathbb{M}y, J_{\mathbb{M}y}^{Gir}) \simeq (\int \mathbb{M}, J_{\mathbb{M}}^{Gir})$$

Stacks from a geometric morphism

Let be $f : \mathcal{E} \to \mathcal{F}$ a geometric morphism. Then we have two indexed categories

$$\begin{array}{ccc} \mathcal{E}^{\mathrm{op}} & \xrightarrow{t_f} & \mathsf{Cat} \\ E & \longmapsto & E \downarrow f^* \end{array} \qquad \begin{array}{ccc} \mathcal{F}^{\mathrm{op}} & \xrightarrow{r_f} & \mathsf{Cat} \\ F & \longmapsto & \mathcal{E} \downarrow f^* F \end{array}$$

where r_f sends $u : F_1 \to F_2$ on the pullback

$$\mathcal{E} \downarrow f^* F_2 \xrightarrow{f^*(u)^*} \mathcal{F} \downarrow f^* F_1$$

Both those indexed categories induce through the Grothendieck construction the same comma category, together with two fibrations

$$\begin{array}{ccccc} & & \mathcal{E} \downarrow f^* & & \\ & \swarrow^{\pi_{t_f}} & & \searrow^{\pi_{r_f}} & \\ \mathcal{E} & & & & \mathcal{F} \end{array}$$

Lifted topology

Definition

The *lifted topology* on $\mathcal{E} \downarrow f^*$ is the topology

$$L_f = \langle J_{t_f}^{Gir} \cup J_{r_f}^{Gir} \rangle$$

jointly generated by the Giraud topologies associated to the canonical topologies on $\mathcal{E}$ and $\mathcal{F}$.

Smallest topology such that we have two comorphisms of (large) sites

$$(\mathcal{E}, J_{can}) \xleftarrow{\ \pi_{t_f}\ } (\mathcal{E} \downarrow f^*, L_f) \xrightarrow{\ \pi_{r_f}\ } (\mathcal{F}, J_{can})$$

Restricting the lifted topology

Size issue: we would need small sites !

When $\mathcal{F} \simeq \mathrm{Sh}(\mathcal{C}_{\mathcal{F}}, J_{\mathcal{F}})$ with $J_{\mathcal{F}}$ generated from a basis $B_{\mathcal{F}}$, we can restrict π_{r_f} to a fibration

$$\mathcal{E} \downarrow f^* y_{\mathcal{C}_{\mathcal{F}}} \xrightarrow{\pi_{r_f y_{\mathcal{C}_{\mathcal{F}}}}} \mathcal{C}_{\mathcal{F}}$$

Then one can restrict the lifted topology to a topology $L_{f,J_{\mathcal{F}}}$ on
f^* sends B-covers to covers in $\mathcal{E}$ for the canonical topology $\mathcal{E} \downarrow f^* y_{\mathcal{C}_{\mathcal{F}}}$

In particular $L_{f,J_{\mathcal{F}}}$ makes both π_{t_f} and $\pi_{r_f y_{\mathcal{C}_{\mathcal{F}}}}$ comorphisms of sites.

Basis for the lifted topology

If one also has a standard site of presentation $(\mathcal{C}_{\mathcal{E}}, J_{\mathcal{E}})$, one can also restrict on the left to $y_{\mathcal{C}_{\mathcal{E}}} \downarrow f^* y_{\mathcal{C}_{\mathcal{F}}}$

The lifted topology also restricts to $y_{\mathcal{C}_{\mathcal{E}}} \downarrow f^* y_{\mathcal{C}_{\mathcal{F}}}$

We have a L_f-dense functor $(y_{\mathcal{C}_{\mathcal{E}}} \downarrow f^* y_{\mathcal{C}_{\mathcal{F}}}) \to (\mathcal{F} \downarrow f^*)$:
$\to$ Provides a small site for the same sheaf topos

Beware that the projection $y_{\mathcal{C}_{\mathcal{E}}} \downarrow f^* y_{\mathcal{C}_{\mathcal{F}}} \to \mathcal{C}_{\mathcal{E}}$ is not a fibration: fibers of basic elements may not be indexed by basic elements. Yet it is still has the cover lifting property.

Basis for the lifted topology

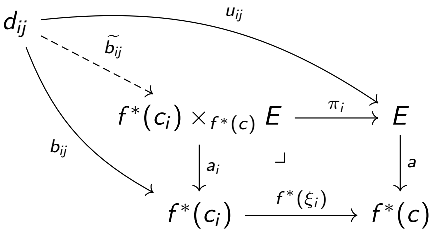
The restricted lifted topology is generated by families of the form

$$((d_{ij}, (c_i, b_{ij})) \xrightarrow{(u_{ij}, (\xi_i, \widetilde{b_{ij}}))} (E, (c, a)))_{i \in I, j \in J_i}$$

where $(\xi_i : c_i \to c)_{i \in I}$ is a family in $\mathcal{B}(c)$ and the families

$$(\widetilde{b_{ij}} : d_{ij} \to f^*(c_i) \times_{f^*(c)} E)_{j \in J_i}$$

are epimorphic in $\mathcal{F}$ for each $i \in I$:



The cartesian stack of a model

If M is a $\mathbb{T}$-model in $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{C}, J)$ corresponding to a $J_{\mathbb{T}}$-flat functor $f_M : \mathcal{C}_{\mathbb{T}} \to \mathcal{E}$, then one has an indexed cartesian category

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{op}} & \xrightarrow{\mathbb{M}} & \mathrm{Cat} \\ c & \longmapsto & (c \downarrow f_M), \\ c_1 \xrightarrow{u} c_2 & \longmapsto & (c_2 \downarrow f_M) \xrightarrow{u^*} (c_1 \downarrow f_M) \end{array}$$

$\mathbb{M}$ is a cartesian stack for J

Each $c \downarrow f_M$ is cartesian as well as the transition functors

The associated comma category is $\int \mathbb{M} \simeq y \downarrow f_M$ which is equipped with a fibration

$$y \downarrow f_M \xrightarrow{\pi_M} \mathcal{C}$$

Antecedent topology

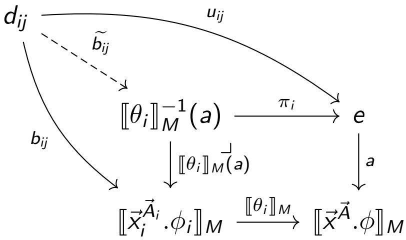
The restriction to $(y \downarrow f_M)$ of the $(f_M, J_{\mathbb{T}})$-lifted topology $L_{(f_M, J_{\mathbb{T}})}$ has as a basis the collection of families

$$\left((d_{ij}\left(\{\vec{x}_i^{\vec{A}_i}.\phi_i\}, [\![\theta_i]\!]_M(a)\right)) \xrightarrow{(u_{ij},([\theta_i]_{\mathbb{T}},\widetilde{b_{ij}}))} (e, (\{\vec{x}^{\vec{A}}.\phi\}, a))\right)_{i\in I, j\in J_i}$$

where for each $i \in I$ we have an epimorphic family

$$\left(\widetilde{b_{ij}} : d_{ij} \to [\![\theta_i]\!]_M^{-1}(a)\right)_{j\in J_i}$$

covering the fibers along the interpretation of $J_{\mathbb{T}}$-covers:



Induced geometric morphism

We have a comorphism of site

$$(y \downarrow f_M, J_M^{Ant}) \xrightarrow{p_M} (\mathcal{C}, J)$$

This induces a geometric morphism

$$\mathrm{Sh}(y \downarrow f_M, J_M^{Ant}) \xrightarrow{u_M} \mathcal{E}$$

We also have a comorphism of site

$$(y \downarrow f_M, J_M^{Ant}) \xrightarrow{\pi_M} (\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$$

The over-topos: general case

Theorem (O.C & A.O.)

For a $\mathbb{T}$-model M in a Grothendieck topos $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{C}, J)$ we have a

$$\mathcal{E}[M] \simeq \mathrm{Sh}(y \downarrow f_M, J_M^{Ant})$$

exhibiting u_M as the over-topos of M.

This means that for any geometric morphism $g : \mathcal{G} \to \mathcal{E}$, any morphism of $\mathcal{E}$-topos

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\quad f \quad} & \mathrm{Sh}(y \downarrow f_M, J_M^{Ant}) \\ & {\scriptstyle g}\searrow \qquad \swarrow{\scriptstyle u_M} & \\ & \mathcal{E} & \end{array}$$

is the name of a morphism of $\mathbb{T}$-models in $\mathbb{T}[\mathcal{G}]$ of the form

$$N \xrightarrow{f} g^* M$$

From geometric morphism to morphism of models

A morphism of $\mathcal{E}$-topos $g \to u_M$ defines a morphism of cartesian stacks over $\mathcal{E}$

$$\mathbb{M} \xrightarrow{f} \mathcal{G}/g^*$$

whose components

$$(c \downarrow F_M) \xrightarrow{f_c} \mathcal{G}/g^* c$$

send basic generalized element $a : c \to [\![\vec{x}^{\vec{A}}.\phi]\!]_M$ to a certain arrow in $\mathcal{G}$

$$N_{(c,a)} \xrightarrow{f_c(a)} g^* c$$

These maps $f_c(a)$ play the same role as the terminal maps $N^{\vec{a}}_{\{\vec{x}^{\vec{A}}.\phi\}} \to 1$ we composed with the $\gamma^*(\vec{a})$ in the set-valued case.

From geometric morphism to morphism of models

Then the strategy is the same as the set-valued case:
We glue the fibers indexed by generalized elements of a given sort

$$N_{\{\vec{x}^{\vec{A}}.\phi\}} := \operatorname*{colim}_{i_{\mathcal{C}} \downarrow [\![\vec{x}^{\vec{A}}.\phi]\!]_M} N_{(c,a)}$$

(but this time as a colimit indexed by all possible basic elements)

Then define f as having as components the induced maps

$$\begin{array}{ccc} N_{(c,a)} & \xrightarrow{f_c(a)} & g^*c \\ {\scriptstyle j_a}\downarrow & & \downarrow{\scriptstyle g^*(a)} \\ N_{\{\vec{x}^{\vec{A}}.\phi\}} & \xrightarrow[f_{\{\vec{x}^{\vec{A}}.\phi\}}]{} & [\![\vec{x}^{\vec{A}}.\phi]\!]_{g^*M} \end{array}$$

Showing those data defines a morphism $f : N \to g^*M$ is technical but mostly use stability of colimits and cancellations of epimorphisms.

From morphisms of models to geometric morphisms

If now one has $f : N \to g^* M$, we just have to consider the pullback of a generalized elements along component of g

$$\begin{array}{ccc} N_{(c,a)} & \longrightarrow & g^* c \\ \downarrow & \lrcorner & \downarrow{\scriptstyle g^*(a)} \\ [\![\vec{x}^{\vec{A}}.\phi]\!]_N & \xrightarrow{f_{\{\vec{x}^{\vec{A}}.\phi\}}} & [\![\vec{x}^{\vec{A}}.\phi]\!]_{g^* M} \end{array}$$

This defines a functor

$$\int \mathbb{M} \xrightarrow{N_{(-)}} \mathcal{G}$$

Proving this functor to be J_M^{Ant}-flat is again done by a stability of epimorphisms.

Last remark on the antecedent topology

When defined from its basic covers, which are dependent on the choice of the presentation $(\mathcal{C}, J)$, J_M^{Ant} has quite complicated presentation.

Moreover restricting to a generator for the base topos breaks the fibration property of the projection.

But when seeing it as induced from the lifting topology $L(f_M, J_{\mathbb{T}})$ it is revealed as something more invariant and meaningful, as the smallest topology making the associated fibrations on $\mathcal{C}_{\mathbb{T}}$ and $\mathcal{C}$ comorphisms.

Perspective

Grothendieck-Verdier formula for localization: for $p : \mathrm{Set} \to \mathcal{E}$ one has a cofiltered 2-limit of etale morphism

$$\mathcal{E}_p \simeq \lim_{(E,a)\in \int p} \mathcal{E}/E$$

Similar formula for the over-topos ? Which replacements for etale geometric morphisms ? (Complete spreads ?)

Also, using the relativised version of geometric theory, a logical account of the general version of the over-topos ?

General properties of the "externalization" functor returning totally connected morphisms

$$\mathbb{T}[\mathcal{E}] \xrightarrow{\mathcal{E}[-]} \mathbf{TotCo}_{\mathcal{E}}$$

Thank you for your attention !