Every Elementary Higher Topos has a Natural Number Object

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Pre-talk Comments

In the interest of time I will skip many details. For more details you can:

- Ask me now or after the talk.
- Ask me later via email, message, Zoom, ...
- See the paper *"Every Elementary Higher Topos has a Natural Number Object"* arXiv:1809.01734 or Theory and Applications of Categories, Vol. 37, 2021, No. 13, pp 337-377.

The Elephant!

Topos theory comes in many flavors!

- **Grothendieck:** Arose in the work of Grothendieck and Bourbaki in the study of algebraic geometry.
- **Elementary:** Arose in the work of Lawvere and Tierney while studying categorical foundations.

Grothendieck toposes are super nice!

Grothendieck toposes have many nice properties:

- Infinite limits and colimits
- Locally Cartesian closed
- **Locally presentable**
- **Giraud's theorem**
- Subobject classifier

Elementary toposes are nice!

Elementary toposes have some of these nice properties:

- ~~In~~**finite limits** and colimits
- **Locally Cartesian closed**
- ~~Locally presentable~~
- (a version of) Giraud's theorem
- **Subobject classifier**

How to recover the magic?

- Less conditions means more examples (finite sets, filter products, realizability toposes), but also less results.
- In particular we lose the ability to construct objects that are “infinite” in nature.
- One way to recover such objects is via **natural number objects**, which come in several flavors.

Freyd natural number object

One elementary way to indicate an object is infinite is to state it has a non-trivial self-injection.

Definition (Freyd NNO)

A triple $(\mathbb{N}, o : 1 \to \mathbb{N}, s : \mathbb{N} \to \mathbb{N})$ is a **Freyd natural number object** if the following are colimits:

$$\mathbb{N} \underset{s}{\overset{id}{\rightrightarrows}} \mathbb{N} \longrightarrow 1 \ , \quad \begin{array}{ccc} \emptyset & \longrightarrow & \mathbb{N} \\ \downarrow & & \downarrow{\scriptstyle s} \\ 1 & \xrightarrow{o} & \mathbb{N} \end{array} \ .$$

Peano natural number object

Another elementary way to define an infinite object is to focus on the set of natural numbers and axiomatize the Peano axioms.

Definition (Peano NNO)

A triple $(\mathbb{N}, o : 1 \to \mathbb{N}, s : \mathbb{N} \to \mathbb{N})$ is a **Peano natural number object** if s is monic, o and s are disjoint subobjects of $\mathbb{N}$, and for every subobject $\mathbb{N}' \hookrightarrow \mathbb{N}$ that is closed under the maps o and s, meaning we have a commutative diagram

$$\begin{array}{ccccc} & \overset{o}{\nearrow} & \mathbb{N}' & \xrightarrow{s} & \mathbb{N}' \\ 1 & & \downarrow & & \downarrow \\ & \underset{o}{\searrow} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \end{array}$$

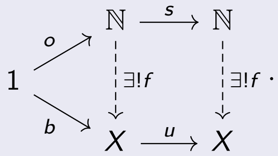
the inclusion $\mathbb{N}' \overset{\cong}{\hookrightarrow} \mathbb{N}$ is an isomorphism.

Lawvere natural number object

Finally, Lawvere found another categorical way to think about NNOs, namely via a universal property.

Definition (Lawvere NNO)

A triple $(\mathbb{N}, o : 1 \to \mathbb{N}, s : \mathbb{N} \to \mathbb{N})$ is a **Lawvere natural number object** if for any other triple $(X, b : 1 \to X, u : X \to X)$ there is a unique map $f : \mathbb{N} \to X$ making the following diagram commute



A Zoo of NNOs

How are all these notions related? We have in fact the best possible result.

Theorem (Elephant D5.1.2)

Let $(\mathbb{N}, o, s)$ be a triple in an elementary topos. Then the following are equivalent.

1. *It is a Freyd natural number object.*
2. *It is a Peano natural number object.*
3. *It is a Lawvere natural number object.*

Hence, we can simply call such an object a natural number object.

Natural Number Objects are Useful

Using natural number objects we now have nice extra results.

Proposition (Elephant D5.3.3)

Let $\mathcal{E}$ be an elementary topos with NNO. Then forgetful functor $\mathcal{M}\mathrm{on}(\mathcal{E}) \to \mathcal{E}$ has a left adjoint given explicitly by

$$F(X) = \mathcal{E}_{/\mathbb{N}}(\mathbb{N}_1, X \times \mathbb{N}),$$

where $\mathbb{N}_1 \to \mathbb{N}$ is the universal finite cardinal.

This condition cannot be relaxed and assuming the existence of NNOs is in fact necessary!

What about higher dimensions?

Let's summarize:

1. We have several notions of natural number objects that all coincide.
2. The existence does not follow from the axioms of elementary toposes.
3. Assuming its existence we can prove cool things.

How about ∞-categories? Can we generalize the results we just reviewed? Are there are any differences?

What is an ∞-Category?

For the purposes of this talk we will take an ∞-category to be the following data:

1. Objects X, Y, ... in $\mathcal{C}$.
2. Mapping **space** $\mathrm{Map}_{\mathcal{C}}(X, Y)$ with a composition operation defined up to contractible ambiguity.
3. All classical categorical terms (limits, adjunctions, Cartesian closure, ...) still hold, although some need to be adjusted.

Which ∞-Categories?

Ideally, we want an **elementary ∞-topos**, which satisfies following conditions:

1. Finite limits and colimits
2. Locally Cartesian closed
3. Subobject classifier
4. Object classifier

There is an ongoing discussion which notion of object classifier to use (how strict, how functorial, how closed it should be).

Which ∞-Categories?

So, instead we will only require the following, strictly weaker conditions:

1. Finite limits and colimits
2. Locally Cartesian closed
3. Subobject classifier
4. **Finite descent**

Remark

Has an underlying *elementary topos* of 0-truncated objects.

Do they exist?

Examples include:

- The ∞-category of spaces.
- Presheaves $\text{Fun}(\mathcal{C}^{op}, \mathcal{S})$
- Grothendieck ∞-toposes
- Filter product ∞-toposes.
- ...

Natural Number Objects in ∞-Categories

Given that we have finite colimits, we can adjust all three definitions for Freyd, Peano and Lawvere natural number objects.

- **Freyd:** We have colimits $\mathbb{N} \underset{s}{\overset{id}{\rightrightarrows}} \mathbb{N} \longrightarrow 1$, $1 \coprod \mathbb{N} \xrightarrow[\cong]{o+s} \mathbb{N}$.
- **Peano:** s is monic, o and s are disjoint subobjects of $\mathbb{N}$, and every subobject $\mathbb{N}' \hookrightarrow \mathbb{N}$ that is closed under the maps o and s is isomorphic to $\mathbb{N}$.
- **Lawvere:** The **space** of maps

$$\begin{array}{ccccc} & \overset{o}{\nearrow} & \mathbb{N} & \xrightarrow{s} & \mathbb{N} \\ 1 & & \downarrow f & & \downarrow f \\ & \underset{b}{\searrow} & X & \xrightarrow{u} & X \end{array} .$$

is **contractible**.

Main Theorem

Theorem (R)

Let $\mathcal{E}$ be a locally Cartesian closed finitely bicomplete ∞-category which satisfies finite descent and has a subobject classifier. Then all three notions of natural number object coincide and exist!

Main Steps of the Proof

The proof neatly breaks down in three major steps each using a different type of mathematics:

1. **Algebraic Topology:** Constructing the free group on one generator generalizing the computation $\pi_1(S^1) = \mathbb{Z}$.
2. **Elementary Topos Theory:** Constructing a Freyd and Peano NNO in $\mathcal{E}$ using results from classical toposes.
3. **Homotopy Type Theory:** Proving it is also a Lawvere NNO in $\mathcal{E}$ using ideas and techniques in homotopy type theory developed by Shulman.

Who cares?

We want to end this talk with some implications:

1. Relation to enveloping ∞-toposes
2. External and Internal Colimits
3. Connections to Truncations
4. Future Directions

Not All Elementary Toposes Lift

Here is a cool implication of these results relating 1-categories and ∞-categories.

Corollary

Let $\mathcal{E}$ be an elementary topos without natural number object (such as finite sets), then there does not exist a locally Cartesian closed finitely bicomplete ∞-category $\hat{\mathcal{E}}$ satisfying finite descent with subobject classifier such that $\tau_0(\hat{\mathcal{E}}) \simeq \mathcal{E}$.

This is in stark contrast to Grothendieck toposes, which always have *enveloping ∞-toposes*.

Implications for Examples of ∞-Toposes

Note this also means that the ∞-category of finite spaces cannot give us ∞-toposes (unlike finite sets). Rather we have to take κ-small spaces for κ large enough. This has also been studied by Lo Monaco.

NNOs and Infinite Colimits

We can use natural number objects and universes to study colimits.

1. Using natural number objects we can reduce the existence of arbitrary countable colimits to the existence of infinite coproduct of the terminal object.
2. More generally we can use natural number objects to define and compute internal sequential limits.

Truncations

The existence of natural number objects and sequential colimits can be used in a variety of ways.

- Using natural number objects we can define internal truncation levels (which can differ from the classical ones in spaces).
- We can translate other homotopy type theory results, such as the join construction (due to Rijke) to get (-1)-truncations.

For more details see *"An Elementary Approach to Truncations"* arXiv:1812.10527.

Further Questions

Can we use natural number objects to construct free A_∞-monoids? We can use the same construction to get the object we expect to be the free A_∞-monoid, however:

1. In the ∞-setting we need operads to define A_∞-monoids.
2. Even more generally the definition of operads could depend on the natural number object.
3. Developing A_∞-objects in homotopy type theory has been challenging.
4. ...

The End

Thank You!

Questions?