Classifying toposes of “geometric” theories

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I. The notion of classifying topos
II. Syntactic categories and classifying toposes
III. Provability and Grothendieck topologies, presheaf-type theories
The notion of classifying topos
This lecture series is devoted to the following theorem:

**Theorem**

(Lawvere, Makkai, Reyes, Joyal, ... building on Grothendieck, Hakim):

Let $\mathcal{T} = \text{first order “geometric” theory.}$  
Then, there exists a topos  

$$\mathcal{E}_\mathcal{T} = \text{“classifying topos of } \mathcal{T} \text{”}$$

endowed with a $\mathcal{T}$-model

$$\mathcal{U}_\mathcal{T} = \text{“universal model of } \mathcal{T} \text{”}$$

such that, for any topos $\mathcal{E}$, the functor

$$\left\{ \begin{array}{c} \text{category of} \\ \text{toposes morphisms} \\
\small f : \mathcal{E} \to \mathcal{E}_\mathcal{T} \\
\end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{category} \\
\text{of } \mathcal{T} \text{-models} \\
in \mathcal{E} \\
\end{array} \right\},$$

$$(f : \mathcal{E} \to \mathcal{E}_\mathcal{T}) \longmapsto f^* \mathcal{U}_\mathcal{T} = \text{pull-back of } \mathcal{U}_\mathcal{T} \text{ along } f$$

is an equivalence of categories.
Remarks:

(i) The couple \((E_T, U_T)\) is uniquely determined by \(T\), up to equivalence.

(ii) \(T\) belongs to the world of logic = formalisation of mathematics.

(iii) \(E_T\) belongs to the world of geometry and topology. For Grothendieck, the notion of topos is the most general notion of space.

(iv) In particular, taking \(E =\) topos of sets

\[
\begin{align*}
\left\{ \text{category of points of } E_T \right\} & \rightarrow \left\{ \text{category of set-theoretic models of } T \right\}
\end{align*}
\]

is an equivalence.

(v) For any topos \(E\), there are infinitely many geometric theories \(T\) such that

\[E \cong E_T.\]

(vi) If \(E_{T_1} \cong E_{T_2}\), \(T_1\) and \(T_2\) are called “Morita equivalent”.

\[\Rightarrow\] Beginning of Olivia Caramello’s theory of “toposes as bridges”.
Basic examples of toposes:

• If $X =$ topological space,
  $\mathcal{E}_X =$ category of sheaves (of sets) on $X$.

Proposition:

(i) Points of $X$ induce points of $\mathcal{E}_X$.
   This is one-to-one if $X$ is “sober”.

(ii) Open subsets of $X$
   correspond to “open subtoposes” of $\mathcal{E}_X$.

Consequence:

→ Toposes generalize topological spaces.
→ They realize an embedding of topology into category theory.

• If $G =$ group,
  $B_G =$ “classifying topos of $G$”
  $=$ category of sets endowed with an action of $G$. 
• If $\mathcal{C}$ = small category,
  $\hat{\mathcal{C}}$ = category of “presheaves” on $\mathcal{C}$
  = category of functors $\mathcal{C}^{\text{op}} \to \text{Set}$.

**Proposition:**

$\hat{\mathcal{C}}$ is a completion of $\mathcal{C}$:

(i) (Yoneda)

$$y : \mathcal{C} \to \hat{\mathcal{C}},$$

$$X \mapsto \text{Hom}(\bullet, X)$$

is fully faithful.

(ii) Any object $P$ of $\hat{\mathcal{C}}$ is a colimit

$$P = \lim_{(X,a) \in \int P} y(X)$$

where $\int P$ = category of “elements of $P$”

$X =$ object of $\mathcal{C}$,

$a \in P(X)$.  

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General definition:
A topos $\mathcal{E}$ is a category which is both a quotient and a subcategory of some $\hat{\mathcal{C}}$ (for $\mathcal{C} =$ small category) in the sense that there are 2 functors such that

\[
(\hat{\mathcal{C}} \xrightarrow{j^*} \mathcal{E}, \mathcal{E} \xleftarrow{j_*} \hat{\mathcal{C}})
\]

- $j^*$ ("sheafification functor") is left-adjoint to $j_*$,
- $j_*$ is fully faithful ($\iff j^* \circ j_* \to \text{id}_\mathcal{E}$ is an isomorphism),
- $j^*$ respects finite limits.

Remarks:

(i) $j_*$ respects arbitrary limits.
(ii) $j^*$ respects arbitrary colimits.
(iii) If $\ell : \mathcal{C} \xrightarrow{y} \hat{\mathcal{C}} \xrightarrow{j^*} \mathcal{E}$, any object $E$ of $\mathcal{E}$ can be written

\[
E = \lim_{(X,a) \in \int j_* E} \ell(X).
\]
Grothendieck topologies:

**Observation:**
If \( X = \text{object of } \mathcal{C} \), a subobject of \( y(X) = \text{Hom}(\bullet, X) \) in \( \hat{\mathcal{C}} \)

\[
S \hookrightarrow y(X)
\]
is a family \( S \) of morphisms \( X' \to X \) stable under composition with any morphism \( X'' \to X' \). This is called a “sieve”.

**Definition:**
- For \( \mathcal{E} = \text{topos presented as } (\hat{\mathcal{C}} \xrightarrow{j^*} \mathcal{E}, \mathcal{E} \xleftarrow{j^*} \hat{\mathcal{C}}) \), a sieve

\[
S \hookrightarrow y(X)
\]
is called “covering” if

\[
(j^* S \hookrightarrow j^* y(X)) = \text{isomorphism in } \mathcal{E}.
\]

- \( J(X) = \text{family of covering sieves of } X \).
- \( J = \text{indexed family of all } J(X), X \in \text{Ob}(\mathcal{C}) \),
  \( = \text{topology on } \mathcal{C} \).
Theorem:

(i) If $J = \text{topology on } \mathcal{C}$, it verifies:

(Maximality) For any $X = \text{object of } \mathcal{C}$,
\[ y(X) \in J(X). \]

(Stability) For any $X' \xrightarrow{f} X$
and $S \in J(X)$,
\[ f^* S = S \times_{y(X)} y(X') \in J(X'). \]

(Transitivity) If $S \in J(X)$
and $S' = \text{sieve of } X$ such that
\[ f^* S' \in J(X') \text{ for any } (X' \xrightarrow{f} X) \in S, \]
then $S' \in J(X)$.

(ii) Conversely, any indexed family of sieves
\[ J = \{ J(X) \mid X = \text{object of } \mathcal{C} \} \]
which verifies those three axioms defines a topos $\mathcal{E} = \widehat{\mathcal{C}_J}$
\[ (\widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}_J}, \widehat{\mathcal{C}_J} \xleftarrow{j^*} \widehat{\mathcal{C}}). \]
Properties of toposes:

= same constructive categorical properties as Set:

- Locally small category.
- Arbitrary colimits and arbitrary limits.
- For any morphism \( E \to S \), the pull-back functor
  \[
  E \times_S \bullet
  \]
  respects arbitrary colimits.
- Sums are disjoint.
- For any object \( E \), quotients
  \[
  E \Rightarrow Q
  \]
  correspond to equivalence relations
  \[
  R \leftrightarrow E \times E
  \]
  by \( R = E \times_Q E, \)
  \[
  Q = \lim_{\to}(R \Rightarrow E).
  \]
- For any object \( E \), subobjects of \( E \) and quotients of \( E \) make up sets.
- If \( u : E' \to E \) is a monomorphism and an epimorphism, \( u = \text{isomorphism} \).
Theorem (Giraud):

Let

\[ \mathcal{E} = \text{category verifying these properties}, \]

\[ \mathcal{C} = \text{small full subcategory of } \mathcal{E} \]
\[ \text{such that any object } X \text{ of } \mathcal{E} \]
\[ \text{has an epimorphic family} \]
\[ X_i \to X \text{ with } X_i = \text{object of } \mathcal{C}, \forall i, \]

\[ J = \text{topology of } \mathcal{C} \]
\[ \text{for which } S \hookrightarrow y(X) \text{ is covering} \]
\[ \text{if it contains an epimorphic family} \]
\[ (X_i \to X). \]

Then

\[ \mathcal{E} \cong \hat{\mathcal{C}}_J \]

is a topos.
Basic examples of morphisms of toposes:

- If

\[ X \xrightarrow{f} Y \]

is a continuous map between topological spaces, it induces a pair of adjoint functors:

\[ \left( \mathcal{E}_Y \xrightarrow{f^*} \mathcal{E}_X , \mathcal{E}_X \xrightarrow{f^*} \mathcal{E}_Y \right) \]

with pull-back \( \parallel \) push-forward.

Furthermore, \( f^* \) respects finite limits.

- If \( J \) = topology on \( C \) = small category,

\[ \left( \widehat{C} \xrightarrow{j^*} \widehat{C}_J , \widehat{C}_J \xleftarrow{j_*} \widehat{C} \right) . \]

- If \( \rho : C \to D \) is a functor between two small categories, it induces:

\[ (\rho^* : \widehat{D} \to \widehat{C}) = \text{composition with } \rho, \]

\( \rho_* = \text{right adjoint of } \rho^*, \)

\( \rho! = \text{left adjoint of } \rho^*. \)

\( \Rightarrow \) Adjoint pair \( (\rho^*, \rho_*) \) such that \( \rho^* \) respects (arbitrary) limits.
General definition:

(i) A morphism of toposes \( f : \mathcal{E}' \rightarrow \mathcal{E} \)

is a pair of adjoint functors

\[ f = (\mathcal{E} \xrightarrow{f^*} \mathcal{E}', \mathcal{E}' \xrightarrow{f_*} \mathcal{E}) \]

whose pull-back component

\( f^* \) respects finite limits.

(ii) A transform of morphisms of toposes

\[ (\mathcal{E}' \xrightarrow{f} \mathcal{E}) \rightarrow (\mathcal{E}' \xrightarrow{g} \mathcal{E}) \]

is a transform of functors

\( f^* \rightarrow g^* \) (or equivalently \( g_* \rightarrow f_* \)).

(iii) The category of points of a topos \( \mathcal{E} \)

is the category of morphisms of toposes

\( \text{Set} \rightarrow \mathcal{E} \).

(iv) An embedding of toposes is a morphism

\[ j = (\mathcal{E} \xrightarrow{j^*} \mathcal{E}', \mathcal{E}' \xrightarrow{j_*} \mathcal{E}) \]

whose push-forward component \( j_* \) is fully faithful.
Remarks:

- Any continuous map between topological spaces
  \[ f : X \to Y \]
  induces a morphism of toposes
  \[(f^* : \mathcal{E}_Y \to \mathcal{E}_X, \ f_* : \mathcal{E}_X \to \mathcal{E}_Y)\].
  This is one-to-one if \( Y \) is sober.
- Any topology \( J \) on \( C \) defines an embedding
  \[ \hat{C}_J \hookrightarrow \hat{C} \].
- Any functor between small categories \( \rho : C \to D \)
  defines a morphism of toposes
  \[(\rho^* : \hat{D} \to \hat{C}, \ \rho_* : \hat{C} \to \hat{D})\]
  whose pull-back component \( \rho^* \) also has a left adjoint
  \[ \rho !_* : \hat{C} \to \hat{D} \].
Languages of first-order theories

Definition:
A first-order language (or signature) $\Sigma$ consists in

- a family of “sorts” ($=$ names of objects)
  
  $G, R, K, V, M, \cdots$

- a family of “function symbols” ($=$ names of morphisms)
  
  $f : A_1 \cdots A_n \to B$ \hspace{1em} (where $A_1, \cdots, A_n, B = \text{sorts}$),

- a family of “relation symbols” ($=$ names of relations)
  
  $R \hookrightarrow A_1 \cdots A_n$ \hspace{1em} (where $A_1, \cdots, A_n = \text{sorts}$).

Remark:
If $n = 0$:

$(f : \to B) = \text{“constant symbol”}$,

$(R \hookrightarrow) = \text{“proposition symbol”}$. 
Examples:

(i) Language of the theory of groups:
- one sort $G$ (= “group”),
- three function symbols

\[
\cdot : GG \to G \quad (\text{multiplication}),
\]

\[
1 : \to G \quad (\text{unit element}),
\]

\[
(\bullet)^{-1} : G \to G \quad (\text{inverse}),
\]
- no relation symbol.

(ii) Language of the theory of equivalence relations
- one sort $E$ (= underlying object),
- no function symbol,
- one relation symbol

\[ R \hookrightarrow EE \quad (= \text{equivalence relation}). \]
Interpretations of signatures

Definition:
Let $\Sigma = \text{signature}$, $\mathcal{E} = \text{topos (or category with finite products, including a terminal object } 1_{\mathcal{E}} \text{)}$.

(i) A $\Sigma$-structure $M$ in $\mathcal{E}$ is a map

\[
\begin{align*}
sort A & \mapsto MA = \text{object of } \mathcal{E}, \\
\left( \text{function symbol } f : A_1 \cdots A_n \to B \right) & \mapsto \left( \text{morphism } MA_1 \times \cdots \times MA_n \xrightarrow{Mf} MB \right) \\
\text{and, if } n = 0, \\
\left( \text{relation symbol } R \twoheadrightarrow A_1 \cdots A_n \right) & \mapsto \left( \text{subobject } MR \hookrightarrow MA_1 \times \cdots \times MA_n \right)
\end{align*}
\]
A morphism of $\Sigma$-structures in $\mathcal{E}$

$$u : M \rightarrow N$$

is a map

sort $A \mapsto \left(\text{morphism of } \mathcal{E}\right)$

such that

- for any $f : A_1 \cdots A_n \rightarrow B$,

$$MA_1 \times \cdots \times MA_n \overset{Mf}{\rightarrow} MB$$

$$u_{A_1} \times \cdots \times u_{A_n} \downarrow \quad \downarrow u_B$$

$$NA_1 \times \cdots \times NA_n \overset{Nf}{\rightarrow} NB$$

is commutative,

- for any $R \rightrightarrows A_1 \cdots A_n$, there is a factorization:

$$MR \xleftarrow{\ } MA_1 \times \cdots \times MA_n$$

$$\downarrow \quad \downarrow u_{A_1} \times \cdots \times u_{A_n}$$

$$NR \xleftarrow{\ } NA_1 \times \cdots \times NA_n$$
Consequences:

\( \Sigma = \) signature.

(i) If \( \mathcal{E} = \) topos

or category with finite products,

\( \Sigma\text{-str}(\mathcal{E}) = \) category

of \( \Sigma \)-structures in \( \mathcal{E} \).

(ii) If

\[ F : \mathcal{E}' \longrightarrow \mathcal{E} \]

= functor which respects finite limits

(or, more generally, which respects finite products and monomorphisms),

there is an induced functor

\[ F : \Sigma\text{-str}(\mathcal{E}') \longrightarrow \Sigma\text{-str}(\mathcal{E}) . \]

(iii) In particular, if

\[ f : \mathcal{E}' \longrightarrow \mathcal{E} \]

is a morphism of toposes, it induces adjoint functors

\[ f^* : \Sigma\text{-str}(\mathcal{E}) \longrightarrow \Sigma\text{-str}(\mathcal{E}') , \]

\[ f_* : \Sigma\text{-str}(\mathcal{E}') \longrightarrow \Sigma\text{-str}(\mathcal{E}) . \]
The notion of “geometric” (first-order) theory:

Definition:

(i) A “geometric” theory consists in

- a signature $\Sigma$,
- a collection of “sequents”
  \[ \varphi \vdash \vec{x} \psi \]
  relying “geometric” formulas of $\Sigma$
  \[ \varphi, \psi \]
  in the same family of variables
  \[ \vec{x} = (x_1^{A_1} \cdots x_n^{A_n}) \text{ (called a “context”) } \]
  associated with sorts of $\Sigma$
  \[ A_1 \cdots A_n. \]

(ii) A “geometric” formula is built from “atomic” formulas using symbols

- $\land = \text{finite conjunction (plus the empty conjunction } \top = \text{“true”)}$,
- $\lor = \text{arbitrary disjunctions (plus the empty disjunction } \bot = \text{“false”)}$,
- $\exists = \text{existential quantifier in part of the variables.}$
An “atomic” formula is deduced from a relation formula \( R(x_1^{A_1} \cdots x_n^{A_n}) \)
(for a relation symbol \( R \rightarrow A_1 \cdots A_n \) of \( \Sigma \))
or an equality formula
\[
(x_1^{A_1} \cdots x_n^{A_n}) = (y_1^{A_1} \cdots y_n^{A_n})
\]
by substitutions of some variables by “terms”.

A term is deduced from an expression \( f_0 = f(x_1^{A_1} \cdots x_n^{A_n}) \)
(for a function symbol \( f : A_1 \cdots A_n \rightarrow B \))
by an inductive process
\[
f_0, f_1, \cdots, f_k
\]
where each \( f_i \) is deduced from
\[
f_{i-1} = f_{i-1}(z_1^{C_1} \cdots z_m^{C_m})
\]
by replacing some variable \( z_i^{C_i} \) by an expression
\[
g(w_1^{D_1} \cdots w_\ell^{D_\ell})
\]
for a function symbol \( g : D_1 \cdots D_\ell \rightarrow C_i \).
Remark:
A (infinitary) first-order theory allows axioms of the form
\[ \phi \vdash_{\vec{x}} \psi \]
on general “first-order formulas”
\[ \varphi, \psi \]
built from atomic formulas with the symbols
\[ \land = \text{arbitrary conjunctions (plus } \top), \]
\[ \lor = \text{arbitrary disjunctions (plus } \bot), \]
\[ \exists = \text{existential quantifier}, \]
\[ \forall = \text{universal quantifier}, \]
\[ \Rightarrow = \text{implication}, \]
\[ \neg = \text{negation}. \]
Interpretation of terms:

Definition:
Let \( \Sigma = \text{signature} \),
\[ f(x_1^{A_1} \cdots x_n^{A_n}) = \text{term with values in a sort } B, \]
\( \mathcal{E} = \text{topos (or a category with finite products)} \)
\( M = \Sigma\text{-structure.} \)

Then \( f(x_1^{A_1} \cdots x_n^{A_n}) \) is interpreted in \( M \) as a morphism
\[ Mf(x_1^{A_1} \cdots x_n^{A_n}) : MA_1 \times \cdots \times MA_n \to MB \]
is an inductive way:

- If \( f = f_0(x_1^{A_1} \cdots x_n^{A_n}) \) is associated with a function symbol
\[ f_0 : A_1 \cdots A_n \to B, \]
\( Mf = Mf_0. \)

- If \( f = f_k(x_1^{A_1} \cdots x_n^{A_n}) \) is deduced from
\[ f_{k-1}(z_1^{C_1} \cdots z_m^{C_m}) \]
by a substitution
\[ z_i^{C_i} = g(w_1^{D_1} \cdots w_{\ell}^{D_\ell}) \]
(for a function symbol \( g : D_1 \cdots D_\ell \to C_i \)),
\( Mf_k \) is the composite of \( Mf_{k-1} \) with the product of \( Mg \) and \( \text{id}_{MC_j}, j \neq i. \)
Interpretation of atomic formulas

Definition:

Let $\Sigma = \text{signature}$,
$\mathcal{E} = \text{topos (or a category with finite limits and smallest subobjects)}$,
$M = \Sigma$-structure,
$\varphi = \varphi(x_1^{A_1} \cdots x_n^{A_n}) = \text{atomic formula}$.

(i) If $\varphi = \top$,
$M\varphi = MA_1 \times \cdots \times MA_n$ is the biggest subobject.

(ii) If $\varphi = \bot$,
$M\varphi = \emptyset_{MA_1 \times \cdots \times MA_n}$ is the smallest (empty)
subobject of $MA_1 \times \cdots \times MA_n$.

(iii) If $\varphi = R(x_1^{A_1} \cdots x_n^{A_n})$ for a relation symbol $R \hookrightarrow A_1 \cdots A_n$, $M\varphi = MR$.

(iv) If $\varphi$ is an equality relation $(x_1^{A_1} \cdots x_n^{A_n}) = (y_1^{A_1} \cdots y_n^{A_n})$,
$M\varphi$ is the diagonal subobject
$MA_1 \times \cdots \times MA_n \hookrightarrow MA_1 \times \cdots \times MA_n \times MA_1 \times \cdots \times MA_n$.

(v) If $\varphi = \varphi_k$ is deduced from $\varphi_{k-1}$ by a substitution
$z_i^{C_i} = g(w_1^{D_1} \cdots w_\ell^{D_\ell})$
the subobject $M\varphi_k$ is deduced from the subobject $M\varphi_{k-1}$
by base change along $Mg : MD_1 \times \cdots \times MD_\ell \rightarrow MC_i$. 
Interpretation of geometric formulas

Definition:
Let \( \Sigma = \) signature,
\( \mathcal{E} = \) topos (or a category with enough structures),
\( M = \Sigma \)-structure,
\( \varphi = \) geometric formula.

(i) If \( \varphi = \varphi(\vec{x}) = \varphi_1(\vec{x}) \land \cdots \land \varphi_k(\vec{x}) \)
and \( \vec{x} = (x_1^{A_1} \cdots x_n^{A_n}) \), the subobject
\[
M\varphi \hookrightarrow MA_1 \times \cdots \times MA_n
\]
is the intersection (= fiber product) of the subobjects
\[
M\varphi_i \hookrightarrow MA_1 \times \cdots \times MA_n.
\]

(ii) If \( \varphi = \varphi(\vec{x}) = \bigvee_{i \in I} \varphi_i(\vec{x}) \), the subobject
\[
M\varphi \hookrightarrow MA_1 \times \cdots \times MA_n
\]
is the union
\[
\lim \left( \bigsqcup_{i,j} M\varphi_i \times MA_1 \times \cdots \times MA_n M\varphi_j \Rightarrow \bigsqcup_i M\varphi_i \right)
\]
of the subobjects
\[
M\varphi_i \hookrightarrow MA_1 \times \cdots \times MA_n.
\]
If \( \varphi = \varphi(\vec{x}) = (\exists \vec{y}) \psi(\vec{x}, \vec{y}) \)

with \( \vec{x} = (x_1^{A_1} \cdots x_n^{A_n}) \), \( \vec{y} = (y_1^{B_1} \cdots y_m^{B_m}) \), the subobject

\[
M\varphi \hookrightarrow MA_1 \times \cdots \times MA_n
\]

is the image

\[
\lim_{\to} (M\psi \times_{MA_1 \times \cdots \times MA_n} M\psi \Rightarrow M\psi)
\]

of the composed morphism

\[
M\psi \hookrightarrow MA_1 \times \cdots \times MA_n \times MB_1 \times \cdots \times MB_m \rightarrow MA_1 \times \cdots \times MA_n.
\]

**Remark:**

Interpretations of geometric formulas

only use finite limits and arbitrary colimits.

They are preserved by arbitrary base change.

They are also preserved by pull-back functors

\[
f^* : E \longrightarrow E'
\]

associated with toposes morphisms

\[
f : E' \longrightarrow E.
\]
Remarks:

(i) One can prove that arbitrary first-order formulas (which also use symbols $\land, \forall, \Rightarrow, \neg$) are interpretable in any topos $\mathcal{E}$. Moreover, their interpretations are always respected by base change.

(ii) But, for a toposes morphism

$$f : \mathcal{E}' \to \mathcal{E},$$

the functor

$$f^* : \Sigma\text{-str}(\mathcal{E}) \to \Sigma\text{-str}(\mathcal{E}')$$

doesn’t respect in general the interpretations of these symbols.
Models of geometric first-order theory:

Definition:
Let \( \Sigma = \text{signature} \),
\( \mathcal{T} = \text{(geometric) first-order theory of signature} \ \Sigma \),
\( \mathcal{E} = \text{topos (or a category with enough structures)} \).

Then:

(i) A \( \Sigma \)-structure in \( \mathcal{E} \)
\begin{equation*}
M
\end{equation*}
is a “\( \mathcal{T} \)-model” if, for any axiom of \( \mathcal{T} \)
\begin{equation*}
\varphi \models \vec{x} \ \psi \quad \text{of context} \quad \vec{x} = (x_1^A \ldots x_n^A) ,
\end{equation*}
the subobjects
\begin{align*}
M\varphi & \hookrightarrow MA_1 \times \cdots \times MA_n , \\
M\psi & \hookrightarrow MA_1 \times \cdots \times MA_n
\end{align*}
verify the inclusion relation
\begin{equation*}
M\varphi \leq M\psi .
\end{equation*}

(ii) A morphism of \( \mathcal{T} \)-models in \( \mathcal{E} \)
\begin{equation*}
M \longrightarrow N
\end{equation*}
is a morphism of the underlying \( \Sigma \)-structures.
Consequences:

(i) For any $\mathcal{E}$, the category of $\mathbb{T}$-models in $\mathcal{E}$

$$\mathbb{T}\text{-mod}(\mathcal{E})$$

is defined as a full subcategory of

$$\Sigma\text{-str}(\mathcal{E}).$$

(ii) If $\mathbb{T}$ is geometric, any toposes morphism

$$(f : \mathcal{E}' \rightarrow \mathcal{E}) = (f^*, f_*)$$

induces a functor

$$f^* : \mathbb{T}\text{-mod}(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{E}').$$

(iii) If $\mathbb{T}$ is geometric and $M = \mathbb{T}$-model in a topos $\mathcal{E}$, there is an induced functor

$$\begin{cases}
(f : \mathcal{E}' \rightarrow \mathcal{E}) & \mapsto f^* M,
\end{cases}$$

category of toposes morphisms

$$\mathcal{E}' \rightarrow \mathcal{E}$$

$$\longmapsto \mathbb{T}\text{-mod}(\mathcal{E}'),$$

category of $\mathbb{T}$-models in $\mathcal{E}'$. 
(iv) If \((\mathcal{E}_T, U_T)\) is such that

\[
(f : \mathcal{E} \to \mathcal{E}_T) \quad \mapsto \quad f^* U_T
\]

is an equivalence of categories for any topos \(\mathcal{E}\), then

\((\mathcal{E}_T, U_T)\)

is uniquely determined up to equivalence.
Lecture II:

Syntactic categories and classifying toposes
What we want to do:

Start from $\mathcal{T} = \text{“geometric” first-order theory.}$

Construct

\[
\begin{align*}
&\begin{cases}
&\text{• a category } \mathcal{C}_T \text{ with enough properties for } \mathcal{T}\text{-mod}(\mathcal{C}_T) \text{ to be defined,} \\
&\text{• a model } M_T \text{ of } \mathcal{T} \text{ in } \mathcal{C}_T, \\
&\text{• a topology } J_T \text{ on } \mathcal{C}_T \\
\end{cases}
\end{align*}
\]

such that, denoting

\[
\begin{align*}
&\begin{cases}
&\text{• } \mathcal{E}_T = \text{quotient topos of } \widehat{\mathcal{C}}_T \text{ by } J_T, \\
&\text{• } U_T = \mathcal{T}\text{-model in } \mathcal{E}_T \text{ image of } M_T \text{ by} \\
\end{cases}
\end{align*}
\]

\[
\ell : \mathcal{C}_T \xrightarrow{y} \widehat{\mathcal{C}}_T \xrightarrow{i^*} \mathcal{E}_T,
\]

the functor

\[
\begin{align*}
&\begin{cases}
&\text{toposes morphisms } \\
&\mathcal{E} \rightarrow \mathcal{E}_T \\
\end{cases} 
\end{align*}
\]

\[
\begin{align*}
&\rightarrow f^* U_T, \\
&\rightarrow \mathcal{T}\text{-mod}(\mathcal{E})
\end{align*}
\]

is an equivalence for any topos $\mathcal{E}$. 

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Geometric categories:

We want $C_T$ to be “geometric” in the following sense:

**Definition:**

(i) A (locally small) category $C$ is geometric if

- it has finite limits,
- in particular, any morphism $p : X \to Y$ defines a functor on categories of subobjects

\[ p^* : \Omega(Y) \to \Omega(X), \]

\[ (S \hookrightarrow Y) \mapsto (S \times_Y X \hookrightarrow X), \]

- this functor has a left adjoint

\[ \exists_p : \Omega(X) \to \Omega(Y) \]

and it commutes with base change:

\[ \begin{array}{ccc}
X' & \xrightarrow{x} & X \\
\downarrow p' & \Box & \downarrow p \\
Y' & \xrightarrow{y} & Y
\end{array} \quad \quad \begin{array}{ccc}
\Omega(X) & \xrightarrow{x^*} & \Omega(X') \\
\downarrow \exists_p & & \downarrow \exists_{p'} \\
\Omega(Y) & \xrightarrow{y^*} & \Omega(Y')
\end{array} \]
• any family of subobjects

\[ S_i \hookrightarrow Y, \quad i \in I, \]

has a union

\[ \bigvee_{i \in I} S_i \hookrightarrow Y \]

such that, for any \( S \hookrightarrow Y \),

\[ \bigvee_{i \in I} S_i \leq S \quad \text{iff} \quad S_i \leq S, \quad \forall \ i, \]

and it commutes with base change:

\[ (X \xrightarrow{p} Y) \Rightarrow p^* \bigvee_{i \in I} S_i = \bigvee_{i \in I} p^* S_i \]

(ii) A functor between geometric categories

\[ F : \mathcal{C} \rightarrow \mathcal{D} \]

is “geometric” if

• it respects finite limits,
• it respects functors \( \exists_p \) in the sense that

\[ \left( \begin{array}{c c c}
S & \hookrightarrow & X \\
\downarrow p & & \downarrow p \\
Y & \hookrightarrow & Y
\end{array} \right) \Rightarrow F(\exists_p S) = \exists_{F(p)} F(S), \]

• it respects arbitrary unions \( \bigvee_{i \in I} \).
Remarks:

(i) For $\Sigma = \text{signature}$, 
\[ C = \text{geometric category}, \]
\[ M = \Sigma\text{-structure in } C, \]
any geometric formula 
\[ \varphi = \varphi(x_1^{A_1} \cdots x_n^{A_n}) = \varphi(\vec{x}) \]
is interpretable as a subobject 
\[ M\varphi(\vec{x}) \hookrightarrow MA_1 \times \cdots \times MA_n. \]

(ii) If $\mathbb{T} = \text{geometric theory}$, 
\[ M \text{ is a } \mathbb{T}\text{-model in } C \]
if and only if, for any axiom of $\mathbb{T}$ 
\[ \varphi \vdash_{\vec{x}} \psi, \]
\[ M \text{ verifies the inclusion relation} \]
\[ M\varphi(\vec{x}) \leq M\psi(\vec{x}). \]

There is an induced full subcategory 
\[ \mathbb{T}\text{-mod}(C) \hookrightarrow \Sigma\text{-str}(C). \]

(iii) Any geometric functor 
\[ F : C \longrightarrow D \]
induces a functor 
\[ F : \mathbb{T}\text{-mod}(C) \longrightarrow \mathbb{T}\text{-mod}(D). \]
Characterization of the syntactic category:

**Theorem:**

Let $T = \text{geometric theory}$.  

(i) There exists a (essentially small) geometric category $C_T$ endowed with a $T$-model $M_T$, such that, for any geometric category $C$, the functor

$$\left\{ \text{geometric functors } \begin{array}{c} C_T \\ F : C_T \to C \end{array} \right\} \longrightarrow T\text{-mod}(C),$$

$$F \mapsto F(M_T)$$

is an equivalence.

(ii) The couple $(C_T, M_T)$ is well defined up to equivalence.
Construction of the syntactic category:

Definition:
Let $\mathbb{T} = \text{geometric theory of signature } \Sigma$.

(i) **Objects of $C_\mathbb{T}$ are geometric formulas of $\Sigma$**

\[ \varphi(\vec{x}) = \varphi(x_1^{A_1} \cdots x_n^{A_n}), \]

considered up to substitution of variables

\[ (x_1^{A_1} \cdots x_n^{A_n}) \leftrightarrow (y_1^{A_1} \cdots y_n^{A_n}). \]

(ii) **Morphisms of $C_\mathbb{T}$**

\[ \varphi(\vec{x}) \rightarrow \psi(\vec{y}) \]

(if $\vec{x} = (x_1^{A_1} \cdots x_n^{A_n})$ and $\vec{y} = (y_1^{B_1} \cdots y_m^{B_m})$ are disjoint) are geometric formulas

\[ \theta(\vec{x}, \vec{y}) \]

that are “provably functional” in the sense that the sequents

\[
\begin{cases}
\theta \vdash_{\vec{x}, \vec{y}} \varphi \land \psi, \\
\varphi \vdash_{\vec{x}} (\exists \vec{y}) \theta(\vec{x}, \vec{y}), \\
\theta(\vec{x}, \vec{y}) \land \theta(\vec{x}, \vec{y}') \vdash_{\vec{x}, \vec{y}, \vec{y}'} \vec{y} = \vec{y}'
\end{cases}
\]

are provable in $\mathbb{T}$,

up to $\mathbb{T}$-provable equivalence of these formulas.
The composite of two morphisms

\[ \theta(\vec{x}, \vec{y}) : \varphi(\vec{x}) \longrightarrow \psi(\vec{y}), \]
\[ \theta'(\vec{y}, \vec{z}) : \psi(\vec{y}) \longrightarrow \chi(\vec{z}) \]

is defined as the provably functional geometric formula

\[ (\exists \vec{y})(\theta(\vec{x}, \vec{y}) \land \theta'(\vec{y}, \vec{z})) : \varphi(\vec{x}) \longrightarrow \chi(\vec{z}). \]

Remarks:

(i) Objects of \( C_T \) only depend on the signature \( \Sigma \) of \( T \).

(ii) Morphisms of \( C_T \) depend on the notion of “\( T \)-provability”.
Definition:
A sequent between geometric formulas of a signature $\Sigma$

$$\varphi \vdash_{\vec{x}} \psi$$

is called “provable” in a theory $\mathcal{T}$ of signature $\Sigma$
if it can be deduced from the axioms of $\mathcal{T}$

$$\varphi_i \vdash_{\vec{x}_i} \psi_i$$

by a combination of the following rules:

(1) Cut rule:

If $\varphi_1 \vdash_{\vec{x}} \varphi_2$ and $\varphi_2 \vdash_{\vec{x}} \varphi_3$, then $\varphi_1 \vdash_{\vec{x}} \varphi_3$.

(2) Identity rule:

$$\top \vdash_{\vec{x}} (f = f)$$

for any term $f$.

(3) Equality rules:

- If $\top \vdash_{\vec{x}} f_1 = f_2$, then $\top \vdash_{\vec{x}} f_2 = f_1$.
- If $\top \vdash_{\vec{x}} f_1 = f_2$ and $\top \vdash_{\vec{x}} f_2 = f_3$, then $\top \vdash_{\vec{x}} f_1 = f_3$. 
(4) **Substitution rules:**

- If $f_1, f_2, f$ are terms and $f'_1, f'_2$ are deduced from $f_1, f_2$ by substitution of $f$ to some variable, [resp. of $f_1, f_2$ to some variable of $f$], then $\top \vdash f_1 = f_2$ implies $\top \vdash f'_1 = f'_2$.

- If $f_1, f_2$ are terms, $R = \text{relation}$, $R_1, R_2$ deduced from $R$ by substitution of $f_1, f_2$ to some variable, then $\top \vdash f_1 = f_2$ implies $R_1 \vdash R_2$ and $R_2 \vdash R_1$. 
(5) Rules of finitary conjunctions:

- $\varphi \vdash \top$ holds for any $\varphi$ in a context $\vec{x}$.
- For any $\varphi, \varphi_1, \cdots, \varphi_k$ in a context $\vec{x}$,

$$\varphi \vdash \varphi_1 \land \cdots \land \varphi_k$$

is equivalent to

$$\varphi \vdash \varphi_i \text{ for any } i, \ 1 \leq i \leq k.$$ 

(6) Rules of infinitary disjunctions:

- $\bot \vdash \varphi$ holds for any $\varphi$ in a context $\vec{x}$.
- For any $\varphi$ and $\varphi_i, i \in I$, in a context $\vec{x}$,

$$\bigvee_{i \in I} \varphi_i \vdash \varphi$$

is equivalent to

$$\varphi_i \vdash \varphi \text{ for any } i \in I.$$
(7) Distributivity rules:

- For any $\phi$ and $\phi_i$, $i \in I$, in a context $\vec{x}$,

$$\phi \land \left( \bigvee_{i \in I} \phi_i \right) \vdash_{\vec{x}} \bigvee_{i \in I} (\phi \land \phi_i)$$

always holds (as well as the converse sequent).

(8) Rules of existential quantification:

- For any $\phi$ in a context $(\vec{x}, \vec{y})$ and any $\psi$ in the context $\vec{x}$,

$$\phi \vdash_{\vec{x}, \vec{y}} \psi$$

is equivalent to

$$(\exists \vec{y}) \phi \vdash_{\vec{x}} \psi.$$ 

(9) Frobenius rule:

- For any $\phi$ in a context $(\vec{x}, \vec{y})$ and any $\psi$ in the context $\vec{x}$

$$(\exists \vec{y}) \phi \land \psi \vdash_{\vec{x}} (\exists \vec{y})(\phi \land \psi)$$

always holds (as well as the converse sequent).
Quotient theories:

**Definition:**

\[ \Sigma \text{ = signature, } \mathcal{T}, \mathcal{T}' \text{ = geometric theories in the signature } \Sigma. \]

(i) \( \mathcal{T}' \) is called a “quotient” of \( \mathcal{T} \) if any geometric sequent of \( \Sigma \)

\[ \varphi \vdash_{\bar{x}} \psi \]

which is provable in \( \mathcal{T} \) is also provable in \( \mathcal{T}' \).

(ii) \( \mathcal{T} \text{ and } \mathcal{T}' \text{ are called “equivalent” if } \mathcal{T}\text{-provable } \iff \mathcal{T}'\text{-provable}. \)

**Remarks:**

(i) If \( \mathcal{T}' \) is a quotient of \( \mathcal{T} \), \( \mathcal{C}_\mathcal{T} \) as a natural functor to \( \mathcal{C}_{\mathcal{T}'} \), with the same objects.

(ii) If \( \mathcal{T}, \mathcal{T}' \) are equivalent,

\[ \mathcal{C}_\mathcal{T} = \mathcal{C}_{\mathcal{T}'}. \]
Subobjects in syntactic categories:

**Proposition:**

Let \( T = \text{geometric theory of signature } \Sigma \),
\[ C_T = \text{syntactic category of } T, \]
\[ \varphi(\vec{x}) = \varphi(x_1^{A_1} \cdots x_n^{A_n}) = \text{object of } C_T = \text{geometric formula of } \Sigma. \]

Then:

(i) **Subobjects of** \( \varphi(\vec{x}) \) **in** \( C_T \) **correspond to geometric formulas**

such that the sequent
\[ \varphi_1(\vec{x}) \]
\[ \varphi_1 \vdash \vec{x} \varphi \]

is \( T \)-provable.

(ii) **Two subobjects of** \( \varphi(\vec{x}) \)
\[ \varphi_1(\vec{x}) \text{ and } \varphi_2(\vec{x}) \]

verify the inclusion relation
\[ \varphi_1(\vec{x}) \leq \varphi_2(\vec{x}) \]

if and only if the sequent
\[ \varphi_1 \vdash \vec{x} \varphi_2 \]

is \( T \)-provable.
The universal model $M_T$ in $C_T$:

**Definition:**

Let $T = \text{geometric theory of signature } \Sigma$, $C_T = \text{syntactic category of } T$.

Then the $\Sigma$-structure $M_T$ in $C_T$ is defined in the following way:

(i) For any sort $A$, $M_T A$ is the object of $C_T$

$$\top(x^A).$$

(ii) For any function symbol of $\Sigma$

$$f : A_1 \cdots A_n \rightarrow B,$$

$M_T f$ is the morphism of $C_T$

$$\top(x_1^{A_1} \cdots x_n^{A_n}) \xrightarrow{x^B = f(x_1^{A_1} \cdots x_n^{A_n})} \top(x^B).$$

(iii) For any relation symbol of $\Sigma$

$$R \hookrightarrow A_1 \cdots A_n,$$

$M_T R$ is the subobject

$$R(x_1^{A_1} \cdots x_n^{A_n}) \hookrightarrow \top(x_1^{A_1} \cdots x_n^{A_n}).$$
Lemma:
For any geometric formula in the signature $\Sigma$

$$\varphi(\bar{x}) = \varphi(x_1^{A_1} \cdots x_n^{A_n}) ,$$

its interpretation in the $\Sigma$-structure $M_T$ of $C_T$

$$M_T \varphi(\bar{x})$$
is the subobject

$$\varphi(\bar{x}) \hookrightarrow \top(\bar{x}) = \top(x_1^{A_1}) \times \cdots \times \top(x_n^{A_n}) = M_T A_1 \times \cdots \times M_T A_n .$$

Corollary:
A geometric sequent of the signature $\Sigma$

$$\varphi \vdash \bar{x} \psi$$
is $\top$-provable if and only if, as subobjects of $\top(\bar{x})$,

$$\varphi(\bar{x}) \leq \psi(\bar{x}) ,$$
i.e. if and only if it is verified by $M_T$.

In particular, $M_T$ is a model of $\top$ in $C_T$. 
Theorem:
Let $T = \text{geometric category}$, 
$C_T = \text{syntactic category of } T$.
Then, for $C = \text{geometric category}$, the functor

$$\left\{ \text{geometric functors } C_T \to C \right\} \to \mathbb{T}\text{-mod}(C),$$

$$(F : C_T \to C) \mapsto F(M_T)$$

is an equivalence, and a reverse equivalence is

$$\mathbb{T}\text{-mod}(C) \to \left\{ \text{geometric functors } C_T \to C \right\},$$

$$(M) \mapsto (F_M : C_T \to C)$$

defined as:

- for any object $\varphi(\vec{x})$ of $C_T$,
  $$F_M(\varphi(\vec{x})) = M\varphi(\vec{x}),$$
- for any morphism of $C_T$
  $$\theta : \varphi(\vec{x}) \xrightarrow{\theta(\vec{x}, \vec{y})} \psi(\vec{y}),$$

$F_M(\theta)$ is the morphism $M\varphi(\vec{x}) \to M\psi(\vec{y})$ whose graph is $M\theta(\vec{x}, \vec{y})$. 
The syntactic topology $J_T$ on $C_T$:

**Definition:**

A sieve on an object 

$$\psi(\vec{y})$$

of $C_T$

is called “$J_T$-covering” if it contains a family of morphisms

$$\theta_i(\vec{x}_i, \vec{y}) : \varphi_i(\vec{x}_i) \longrightarrow \psi(\vec{y}), \quad i \in I,$$

whose union of images is the full object, equivalently, such that the sequent

$$\psi(\vec{y}) \vdash \bigvee_{i \in I} (\exists \vec{x}_i) \theta_i(\vec{x}_i, \vec{y})$$

is $T$-provable.

**Remarks:**

1. $J_T$ is a topology because the condition for the union of images to be the full object is preserved by base change in $C_T$.
2. $J_T$ is defined by the categorical structure of $C_T$. 

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Diaconescu’s equivalence:

**Theorem:**

Let $\mathcal{C} = \text{(essentially) small category with finite limits}$, 
$J = \text{topology on } \mathcal{C}$,

$$(\ell : \mathcal{C} \xrightarrow{y} \hat{\mathcal{C}} \xrightarrow{j^*} \hat{\mathcal{C}}_J) = \text{canonical functor},$$

$\mathcal{E} = \text{topos}.$

Then:

(i) For any toposes morphism

$$f = (f^*, f_*) : \mathcal{E} \longrightarrow \hat{\mathcal{C}}_J,$$

$$(f^* \circ \ell : \mathcal{C} \rightarrow \mathcal{E}) \text{ is}$$

- “flat” in the sense that it respects finite limits,
- “$J$-continuous” in the sense that it transforms $J$-covering families of $\mathcal{C}$ into globally epimorphic families of $\mathcal{E}$.

(ii) The functor

$$\left\{ \begin{array}{c}
\text{category of toposes morphisms} \\
\mathcal{E} \rightarrow \hat{\mathcal{C}}_J
\end{array} \right\} \longrightarrow 
\left\{ \begin{array}{c}
\text{category of flat } J \text{-continuous functors} \\
\mathcal{C} \rightarrow \mathcal{E}
\end{array} \right\}$$

is an equivalence.
Lemma:

Let
\[ T = \text{geometric theory}, \]
\[ C_T = \text{associated syntactic category}, \]
\[ E = \text{topos}. \]

Then a functor
\[ F : C_T \longrightarrow E \]

is “geometric” if and only if it is

- flat,
- \( J_T \)-continuous,

i.e. transforms families of morphisms whose union of images is the full object into globally epimorphic families of \( E \).
Corollary:

Let

\[ T = \text{geometric theory}, \]
\[ \mathcal{E}_T = \text{quotient topos of } \hat{\mathcal{C}}_T \text{ by } J_T, \]
\[ \mathcal{E} = \text{topos}. \]

Then the composite functor

\[
\begin{align*}
\left\{ \text{category of toposes morphisms} \right\} & \quad \longrightarrow \quad \left\{ \text{flat } J\text{-continuous functors} \right\} \\
\mathcal{E} \to \mathcal{E}_T & \quad \longrightarrow \quad \mathcal{C}_T \to \mathcal{E} \\
\left\{ \text{geometric functors} \right\} & \quad \longrightarrow \quad T\text{-mod}(\mathcal{E})
\end{align*}
\]

is an equivalence of categories.
Definition:

Let
\[ C = \text{(essentially) small category,} \]
\[ J = \text{topology on } C, \]
\[ (\ell : C \to \hat{C} \xrightarrow{j^*} \hat{C}_J) = \text{canonical functor.} \]

Then J is called “subcanonical” if and only if it verifies the following equivalent conditions:

1. The functor \( \ell : C \to \hat{C}_J \)

   is fully faithful.

2. The functor \( y : C \to \hat{C} \)

   factorises through \( j_* : \hat{C}_J \to \hat{C} \).
**Lemma:** Let

\( T = \text{geometric category}, \)

\( C_T = \text{syntactic category of } T, \)

\( E_T = \overline{(C_T)}_{J_T} = \text{classifying topos of } T, \)

\( (\ell : C_T \to E_T) = \text{canonical functor}. \)

Then \( \ell \) is fully faithful i.e. \( J_T \) is subcanonical.

**Corollary:**

A geometric sequent of \( T \)

\[ \varphi \vdash x \psi \]

is \( T \)-provable if and only if

it is verified by the universal model of \( T \)

\[ U_T = \ell(M_T) \quad \text{in} \quad E_T. \]

**Consequence:**

\[ \begin{pmatrix} \text{Gödel’s completeness theorem} \\ \text{Deligne’s theorem on “coherent” toposes} \end{pmatrix} = \begin{pmatrix} \text{Deligne’s theorem on “coherent” toposes having enough points} \end{pmatrix}. \]
**Proposition:**

For any topos $\mathcal{E}$, there are (infinitely many) geometric theories $\mathbb{T}$ such that

$$\mathcal{E}_\mathbb{T} \cong \mathcal{E}.$$ 

**Hint:**

Write $\mathcal{E} \cong \widehat{\mathcal{C}_J}$

for $\mathcal{C} = \text{small category with finite limits}$,

$J = \text{topology on } \mathcal{C}.$

Then define $\mathbb{T}$ as the “geometric theory”

of “flat $J$-continuous” functors on $\mathcal{C}$. 
Definition:

Geometric theories $\mathbb{T}_1$ and $\mathbb{T}_2$ are called “Morita equivalent” or “semantically equivalent” if

$$\mathcal{E}_{\mathbb{T}_1} \cong \mathcal{E}_{\mathbb{T}_2}.$$

Remarks:

(i) The condition

$$\mathcal{E}_{\mathbb{T}_1} \cong \mathcal{E}_{\mathbb{T}_2}$$

is verified if $\mathbb{T}_1$ and $\mathbb{T}_2$ are “syntactically equivalent” in the sense that their syntactic categories are equivalent

$$\mathcal{C}_{\mathbb{T}_1} \cong \mathcal{C}_{\mathbb{T}_2}.$$

(ii) The converse is not true.
Lecture III:

Provability and Grothendieck topologies, presheaf-type theories
The notion of subtopos:

Definition:

(i) A morphism of toposes

\[ j = (j^*, j_*) : \mathcal{E}' \to \mathcal{E} \]

is called an “embedding”

if \( j_\ast \) is fully faithful or, equivalently,

\( j^* \circ j_\ast \to \text{id}_{\mathcal{E}'} \) is an isomorphism.

(ii) A subtopos of a topos \( \mathcal{E} \)

is an equivalence class of embeddings

for the relation defined by diagrams

\[ \mathcal{E}_1 \xrightarrow{j_1} \mathcal{E} \xleftarrow{j_2} \mathcal{E}_2 \]

with \( j_2 \circ e \simeq j_1 \).
Subtoposes and topologies:

If $\mathcal{E} = \widehat{\mathcal{C}}_J = \text{topos associated to } \left\{ \begin{array}{l} \mathcal{C} = \text{category,} \\ J = \text{topology on } \mathcal{C}, \end{array} \right.$

any topology $J' \supseteq J$ on $\mathcal{C}$ defines a subtopos

$$\widehat{\mathcal{C}}_J', \hookrightarrow \widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}.$$

Conversely:

**Proposition:**

If $\mathcal{E} = \widehat{\mathcal{C}}_J$, the map

$$\left( J' \supseteq J \right) \mapsto (\widehat{\mathcal{C}}_J', \hookrightarrow \widehat{\mathcal{C}}_J = \mathcal{E})$$

is one-to-one.

**Remark:**

This implies that, for any topos $\mathcal{E}$, subtoposes of $\mathcal{E}$ make up a set ordered by inclusion.
Subtoposes and quotient theories:

Theorem (Caramello):

Let $T = \text{geometric theory of signature } \Sigma$.

(i) For any $T' = \text{quotient theory of } T$

(= geometric theory of same signature $\Sigma$ with more provable sequents),
$E_{T'}$ is a subtopos of $E_T$.

(ii) The map

\[
\begin{align*}
\left\{ \text{quotient theories } T' \text{ of } T, \\
\text{considered up to equivalence} \right\} & \quad \longrightarrow \\
\left\{ \text{subtoposes } E' \hookrightarrow E_T \right\},
\end{align*}
\]

$T' \longmapsto (E_{T'}, \hookrightarrow E_T)$

is one-to-one.
Sketch of the proof: One has to prove a correspondence:

\[
\begin{align*}
\left\{ \text{quotients } T' \text{ of } T \right\} & \leftrightarrow \left\{ \text{topologies } J \supseteq J_T \text{ on } C_T \right\} \\

\end{align*}
\]

- From a quotient \( T' \) of \( T \) to a topology \( J \) on \( C_T \):

For any axiom of \( T' \),

\[ \varphi \vdash \bar{x} \psi \]

consider the associated monomorphism of \( C_T \)

\[ \varphi \land \psi(\bar{x}) \hookrightarrow \varphi(\bar{x}) \]

and decide it is \( J \)-covering.

Define \( J \) as the smallest topology on \( C_T \) containing \( J_T \) and these coverings.

- From a topology \( J \supseteq J_T \) on \( C_T \) to a quotient \( T' \) of \( T \):

For any \( J \)-covering family of morphisms of \( C_T \)

\[ \theta_i(\bar{x}_i, \bar{x}) : \varphi_i(\bar{x}_i) \longrightarrow \varphi(\bar{x}) \], \quad i \in I,

decide that

\[ \varphi(\bar{x}) \vdash \bar{x} \bigvee_{i \in I} (\exists \bar{x}_i) \theta_i(\bar{x}_i, \bar{x}) \]

is an axiom of \( T' \). Define \( T' \) by the collection of these axioms.
Consequence: provability and Grothendieck topologies

**Corollary:**

Let \( T = \text{geometric theory}, \)
\[ \mathcal{E}_T = \text{classifying topos of } T \text{ written as } \mathcal{E}_T \cong \widehat{C}_J. \]

Then there is a natural (constructible) one-to-one correspondence:
\[
\left\{ \text{quotient } T' \text{ of } T, \text{ up to equivalence} \right\} \leftrightarrow \left\{ \text{topologies } J' \supseteq J \text{ on } C \right\}
\]

**Remark:** This is the bridge:

So, a quotient theory \( T' \) of \( T \) is contradictory if and only if \( J' \) is maximal (i.e. the empty sieve is a \( J' \)-covering of any object of \( C \)).
Topological translation of provability:

Let
\[ T = \text{geometric theory of signature } \Sigma, \]
\[ (\varphi \vdash x \psi) = \text{geometric sequent of } \Sigma, \]
\[ T' = \text{quotient theory of } T \text{ defined by adding the axiom } \varphi \vdash x \psi. \]
\[ \mathcal{E}_T = \text{classifying topos of } T \text{ written as } \mathcal{E}_T \cong \hat{C}_J, \]
\[ (J' \supseteq J) = \text{topology on } C \text{ such that } \hat{C}_{J'} = \mathcal{E}_{T'}. \]

Then we have equivalences:

\[ (\varphi \vdash x \psi) \text{ is provable in } T \]

\[ \Downarrow \]

\[ T' \text{ is equivalent to } T \]

\[ \Downarrow \]

\[ J' = J. \]
How to compute the classifying topos of a theory?

Given a geometric theory $\mathbb{T}$ of signature $\Sigma$,

how to construct $\begin{cases} 
\text{a (essentially) small category } \mathcal{C}, \\
\text{a topology } J \text{ on } \mathcal{C}, 
\end{cases}$

such that

$$\mathcal{E} \cong \hat{\mathcal{C}}_J ?$$

Remarks:

(i) For any $\mathcal{E}$, the collection of equivalences

$$\mathcal{E} \cong \hat{\mathcal{C}}_J$$

is so big that it is not even a set \ldots

(ii) One can always take

$$\begin{cases} 
\mathcal{C} = \mathcal{C}_\mathbb{T} = \text{syntactic category}, \\
J = J_\mathbb{T} = \text{syntactic topology}, 
\end{cases}$$

but, in general, it cannot be made fully explicit.
Presheaf-type theories:

**Definition:**
A geometric theory $\mathbb{T}$
is called “of presheaf type”
if its classifying topos $\mathcal{E}_\mathbb{T}$
can be written as a presheaf topos

$$\mathcal{E}_\mathbb{T} \cong \hat{\mathcal{C}}$$

for some $\mathcal{C} = (\text{essentially}) \text{ small category}$.

**Examples:**

(i) If $\mathbb{T} =$ empty theory (no axiom)
on some signature $\Sigma$,
$\mathbb{T}$ is of presheaf-type.

(ii) $\mathbb{T}$ is of presheaf-type if it is algebraic, i.e.

\[
\left\{ \begin{array}{l}
\text{no relation symbol,} \\
\text{all axioms have the form } \mathbb{T} \vdash \vec{x} f(\vec{x}) = g(\vec{x}) \text{ for } f, g = \text{terms.}
\end{array} \right.
\]
Presheaf toposes:

Theorem:

Let $\mathcal{E} = \hat{\mathcal{C}}$ for $\mathcal{C} = (\text{essentially})$ small category. Then:

(i) **The category of points of $\hat{\mathcal{C}}$** is equivalent to the category of functors $P : \mathcal{C} \rightarrow \text{Set}$ which verify the following equivalent properties:

(1) The category of “elements” of $P$

$$\int P = \{ (X, x) \mid X = \text{object of } X, \ x \in P(X) \}$$

is filtering.

(2) $P$ can be written as a filtering colimit of representable functors $\text{Hom}(X, \bullet)$, $X = \text{object of } \mathcal{C}$.

(ii) **As a consequence,**

$$\text{pt}(\hat{\mathcal{C}}) \cong \text{Ind}(\mathcal{C}^{\text{op}}).$$
**Corollary:**

If $\mathbb{T} = \text{presheaf-type theory}$ with $\mathcal{E}_\mathbb{T} \cong \hat{\mathcal{C}}$, $\mathbb{T}\text{-mod}(\text{Set}) \cong \text{Ind}(\mathcal{C}^{\text{op}})$.

**Remark:** This is the bridge:

points of $\mathcal{E}_\mathbb{T} \cong \hat{\mathcal{C}}$

**Remark:**

So, the category $\text{Ind}(\mathcal{C}^{\text{op}})$ is determined by $\mathbb{T}$, up to equivalence.

→ **Question:** To which extent is it possible to recover $\mathcal{C}$?
**Definition:**

Let $\mathbb{T} = \text{presheaf-type theory.}$

A set-based model $M$ of $\mathbb{T}$ (= object of $\mathbb{T}\text{-mod}(\text{Set})$) is called “finitely presentable” if the functor

$$\text{Hom}(M, \bullet) : \mathbb{T}\text{-mod}(\text{Set}) \to \text{Set}$$

respects filtering colimits.

**Remark:**

One can introduce

$$\mathbb{T}\text{-mod}(\text{Set})_{fp}$$

= full subcategory of $\mathbb{T}\text{-mod}(\text{Set})$
on finitely presentable models.
Proposition:

If $\mathcal{E}_\mathbb{T} \cong \hat{\mathcal{C}}$
and so $\mathbb{T}$-mod(Set) $\cong \text{Ind}(\hat{\mathcal{C}}^{\text{op}})$,

an object $M$ of $\text{Ind}(\hat{\mathcal{C}}^{\text{op}})$
is finitely presentable if and only if there exist

\[
\begin{align*}
\text{an object } X \text{ of } \mathcal{C}^{\text{op}}, \\
a \text{ pair of morphisms } M & \xleftarrow{p} \xrightarrow{i} \text{Hom}(X, \bullet) \text{ with } p \circ i = \text{id}_M.
\end{align*}
\]

Corollary:

If $\mathcal{E}_\mathbb{T} \cong \hat{\mathcal{C}}$, then:

(i) $\mathbb{T}$-mod(Set)$_{fp}$ = Karoubi completion of $\mathcal{C}^{\text{op}}$.

(ii) $\mathcal{E}_\mathbb{T} \cong \widehat{\mathcal{M}}^{\text{op}}$ for $\mathcal{M} = \mathbb{T}$-mod(Set)$_{fp}$.

(iii) Any (set-based) model of $\mathbb{T}$
is a filtering colimit of finitely presentable models.
Consider a geometric theory $\mathbb{T}$ on a signature $\Sigma$.

→ Construct a presheaf-type theory $\mathbb{T}'$ such that $\mathbb{T}$ is a quotient of $\mathbb{T}'$.
   
   Ex: Take for $\mathbb{T}'$ the empty theory on $\Sigma$.
   
   (Warning: It is not always a good choice.)

→ Consider the (essentially) small category

$$M = \mathbb{T}'\text{-mod(Set)}_{fp}$$

with $\mathcal{E}_{\mathbb{T}'} \cong \widehat{M}^{\text{op}}$.

→ Compute the (unique) topology $J$ on $M^{\text{op}}$ such that

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{(M^{\text{op}})}_J.$$
The problem of characterizing presheaf-type theories:

If we want to compute the classifying topos $\mathcal{E}_T$ of some geometric theory $T$, one tries to write $T$ as a quotient of a geometric theory $T'$ which, hopefully, is presheaf-type.

→ **Natural question:** How to prove that a geometric $T'$ is presheaf-type?
The syntactic characterization of presheaf-type theories

For any geometric theory \( \mathcal{T} \), its classifying topos \( \mathcal{E}_\mathcal{T} \) is associated to

\[
\begin{align*}
\mathcal{C}_\mathcal{T} &= \text{syntactic category of } \mathcal{T}, \\
\mathcal{J}_\mathcal{T} &= \text{syntactic topology of } \mathcal{C}_\mathcal{T}
\end{align*}
\]

and the canonical functor

\[
\ell : \mathcal{C}_\mathcal{T} \hookrightarrow \widehat{\mathcal{C}}_\mathcal{T} \xrightarrow{j^*} \mathcal{E}_\mathcal{T}
\]

is fully faithful.

Furthermore:

\[
\begin{align*}
\bullet \ & \text{objects of } \mathcal{C}_\mathcal{T} = \text{geometric formulas } \varphi(\vec{x}) \text{ on the signature } \Sigma \text{ of } \mathcal{T}, \\
\bullet \ & \text{morphisms of } \mathcal{C}_\mathcal{T} = \text{geometric formulas } \\
\theta(\vec{x}, \vec{y}) : \varphi(\vec{x}) & \longrightarrow \psi(\vec{y})
\end{align*}
\]

which are \( \mathcal{T} \)-provably functional, up to \( \mathcal{T} \)-provable equivalence,

\( \bullet \ \mathcal{J}_\mathcal{T} \)-covering families = globally epimorphic families.
Definition:
(i) An object $E$ of a topos $\mathcal{E}$ is called “irreducible” if any globally epimorphic family
$$E_i \rightarrow E, \quad i \in I,$$
has a splitting
$$E \rightarrow E_{i_0} \quad \text{for some } i_0 \in I$$
such that $(E \rightarrow E_{i_0} \rightarrow E) = \text{id}_E$.

(ii) If $\{C = (\text{essentially}) \text{ small category},
J = \text{topology on } C,$
an object $X$ of $C$
is called “irreducible”
if its only $J$-covering sieve is the maximal sieve.

(iii) If $\mathbb{T} = \text{geometric theory of signature } \Sigma,$
a geometric formula of $\Sigma$
is called “irreducible” if
it is irreducible as an object of $\mathcal{C}_\mathbb{T}$
for the topology $J_\mathbb{T}$. 
Remark:
A geometric formula of $\Sigma$

$$\varphi(\vec{x})$$

is $J_T$-irreducible in $C_T$ if and only if:

For any family of $T$-provably functional formulas

$$\varphi_i(\vec{x}_i) \xrightarrow{\theta_i(\vec{x}_i, \vec{x})} \varphi(\vec{x}), \quad i \in I,$$

such that $\varphi(\vec{x}) \not\vdash \bigvee_{i \in I} (\exists \vec{x}_i)(\theta_i(\vec{x}_i, \vec{x})$ is $T$-provable,

there exist

\begin{align*}
\begin{cases}
\text{• an index } i_0 \in I, \\
\text{• a } T\text{-provably functional formula}
\end{cases}
\end{align*}

$$\varphi(\vec{x}) \xrightarrow{\theta(\vec{x}, \vec{x}_{i_0})} \varphi_{i_0}(\vec{x}_{i_0})$$

such that the formula

$$(\exists \vec{x}_{i_0})(\theta(\vec{x}, \vec{x}_{i_0}) \land \theta_{i_0}(\vec{x}_{i_0}, \vec{x}'))$$

is $T$-provably equivalent to the formula

$$\vec{x} = \vec{x}'.$$
Lemma: Considering the bridge

irreducible objects of
$\mathcal{E}_T \cong \widehat{\mathcal{C}}$

we have:

(i) If $\mathcal{E} = \widehat{\mathcal{C}}$, irreducible objects of $\mathcal{E}$ are splittings of representable objects $\text{Hom}(\bullet, X), X = \text{object of } \mathcal{C}$.

(ii) If $\mathcal{E} = \mathcal{E}_T$, irreducible objects of $\mathcal{E}$ are irreducible formulas $\varphi(\vec{x})$.

Remarks:

(i) If $\mathcal{E} = \widehat{\mathcal{C}}$, the Karoubi completion of $\mathcal{C}$ is the full subcategory of $\mathcal{E}$ on irreducible objects.

(ii) If $\mathcal{T}$ is presheaf-type, there is a one-to-one correspondence

\[
\begin{align*}
\{ \text{irreducible formulas} \} & \leftrightarrow \{ \text{finitely presentable set-based models of } \mathcal{T} \}.
\end{align*}
\]
Theorem (Caramello):

Let $\mathbb{T} = \text{geometric theory on a signature } \Sigma$.

Then $\mathbb{T}$ is presheaf-type if and only if

any geometric formula of $\Sigma$

considered as an object of $\mathcal{C}_\mathbb{T}$, has a $J_\mathbb{T}$-covering

by irreducible formulas

$$\theta_i(\bar{x}_i, \bar{y}) : \varphi_i(\bar{x}_i) \rightarrow \psi(\bar{y})$$
Another criterion for being presheaf-type: the equivalence of syntax and semantics

Theorem (Caramello):

Let $\mathbb{T} = \text{geometric theory on a signature } \Sigma$.

Then $\mathbb{T}$ is presheaf-type if and only if the following 3 conditions are verified:

1. For any geometric sequent of $\Sigma$

   \[ \varphi \vdash_{\chi} \psi, \]

   it is $\mathbb{T}$-provable if and only if it is verified by all set-based models.
Any finitely presentable set-based model

\[ M = \text{object of } T\text{-mod(}\text{Set})\text{}_{fp} \]

is finitely presented
by some (irreducible) geometric formula

\[ \varphi(x_1^{A_1}, \ldots, x_n^{A_n}) \]

in the sense that,
for any set-based model \( N \),
there is a natural one-to-one correspondence

\[
\begin{align*}
\{ \text{model morphisms} \} & \quad \leftrightarrow \quad \{ \text{families of elements} \} \\
M \to N & \quad \leftrightarrow \quad (x_1^{A_1}, \ldots, x_n^{A_n}) \in N\text{A}_1 \times \cdots \times N\text{A}_n \\
\text{which verify the formula} & \quad \varphi(x_1^{A_1}, \ldots, x_n^{A_n}) \).
\end{align*}
\]
For any sorts $A_1, \cdots, A_n$ of $\Sigma$ and any map

$$P : M \mapsto MP$$

finitely presentable subset of set-based model of $T$

such that, any model morphism

$$u : M \to N$$

induces a commutative square

then $P$ is defined by a geometric formula

$$\varphi(x_1^{A_1} \cdots x_n^{A_n}).$$
Proof in the direct sense:

(1) follows from the equivalence

\[ (\widehat{\mathcal{C}_T})_{T} \cong \mathcal{E}_T \cong \mathcal{M}^{\text{op}} \]

with \( \mathcal{M} = T\text{-mod}(\text{Set})_{fp} \).

(2) follows from the bridge:

irreducible objects of

\[ \mathcal{E}_T \cong \mathcal{M}^{\text{op}} \]

(3) follows from the bridge

subobjects of

\[ U_T A_1 \times \cdots \times U_T A_n \]

(for \( U_T = \text{universal model of } T \text{ in } \mathcal{E}_T \)).
Application to the computation of classifying toposes:

Suppose that $T'$ is a quotient of $T$, $T$ is a presheaf-type theory and so

$$E_T \cong \hat{M}^{\text{op}}$$

for $M = T\text{-mod (Set)}_{fp}$,

$M^{\text{op}} \cong$ full subcategory of $C_T$ on irreducible formulas.

**Corollary:** We have

$$E_{T'} \cong (\hat{M}^{\text{op}})_J$$

if $J$ is the topology on $M^{\text{op}}$ defined in the following way:

A family of morphisms between irreducible formulas

$$\theta_i(\bar{x}_i, \bar{x}) : \varphi_i(\bar{x}) \rightarrow \varphi(\bar{x}), \quad i \in I,$$

(corresponding to a family of morphisms between finitely presented models

$$M \rightarrow M_i, \quad i \in I$$)

is $J$-covering if and only if the sequent

$$\varphi(\bar{x}) \vdash \bigvee_{i \in I} (\exists \bar{x}_i) \theta_i(\bar{x}_i, \bar{x})$$

is $T'$-provable.