Classifying toposes of “geometric” theories

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I. The notion of classifying topos

II. Syntactic categories and classifying toposes

III. Provability and Grothendieck topologies, presheaf-type theories

Lecture I:

The notion of classifying topos

This lecture series is devoted to the following theorem:

Theorem

(Lawvere, Makkai, Reyes, Joyal,... building on Grothendieck, Hakim):

Let $\mathbb{T}$ = first order "geometric" theory.
Then, there exists a topos

$$\mathcal{E}_{\mathbb{T}} = \text{"classifying topos of } \mathbb{T}\text{"}$$

endowed with a $\mathbb{T}$-model

$$U_{\mathbb{T}} = \text{"universal model of } \mathbb{T}\text{"}$$

such that, for any topos $\mathcal{E}$, the functor

$$\left\{\begin{array}{c}\text{category of}\\ \text{toposes morphisms}\\ f : \mathcal{E} \to \mathcal{E}_{\mathbb{T}}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{category}\\ \text{of } \mathbb{T}\text{-models}\\ \text{in } \mathcal{E}\end{array}\right\},$$

$$(f : \mathcal{E} \to \mathcal{E}_{\mathbb{T}}) \longmapsto f^* U_{\mathbb{T}} = \text{pull-back of } U_{\mathbb{T}} \text{ along } f$$

is an equivalence of categories.

Remarks:

(i) The couple $(\mathcal{E}_{\mathbb{T}}, U_{\mathbb{T}})$ is uniquely determined by $\mathbb{T}$, up to equivalence.

(ii) $\mathbb{T}$ belongs to the world of logic $=$ formalisation of mathematics.

(iii) $\mathcal{E}_{\mathbb{T}}$ belongs to the world of geometry and topology.
For Grothendieck, the notion of topos is the most general notion of space.

(iv) In particular, taking $\mathcal{E} =$ topos of sets

$$\left\{\begin{array}{c}\text{category of}\\ \text{points of } \mathcal{E}_{\mathbb{T}}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{category of}\\ \text{set-theoretic models of } \mathbb{T}\end{array}\right\}$$

is an equivalence.

(v) For any topos $\mathcal{E}$,
there are infinitely many geometric theories $\mathbb{T}$ such that

$$\mathcal{E} \cong \mathcal{E}_{\mathbb{T}}\,.$$

(vi) If $\mathcal{E}_{\mathbb{T}_1} \cong \mathcal{E}_{\mathbb{T}_2}$,
$\mathbb{T}_1$ and $\mathbb{T}_2$ are called “Morita equivalent”.

$\Rightarrow$ Beginning of Olivia Caramello’s theory of “toposes as bridges”.

Basic examples of toposes:

- If X = topological space,
  $\mathcal{E}_X$ = category of sheaves (of sets) on X.

Proposition:

(i) *Points of X induce points of $\mathcal{E}_X$.*
This is one-to-one if X is "sober".

(ii) *Open subsets of X*
correspond to "open subtoposes" of $\mathcal{E}_X$.

Consequence:

$\rightarrow$ Toposes generalize topological spaces.

$\rightarrow$ They realize an embedding
of topology into category theory.

- If G = group,
  B_G = "classifying topos of G"
  = category of sets endowed with an action of G.

• If $\mathcal{C}$ = small category,
$\widehat{\mathcal{C}}$ = category of "presheaves" on $\mathcal{C}$
= category of functors

$$\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}.$$

Proposition:

$\widehat{\mathcal{C}}$ is a completion of $\mathcal{C}$:

(i) *(Yoneda)*

$$\begin{array}{cccc} y & : & \mathcal{C} & \longrightarrow & \widehat{\mathcal{C}}, \\ & & X & \longmapsto & \mathrm{Hom}(\bullet, X) \end{array}$$

is fully faithful.

(ii) *Any object P of $\widehat{\mathcal{C}}$ is a colimit*

$$P = \varinjlim_{(X,a)\in\int P} y(X)$$

where $\int P$ = category of "elements of P"
X = object of $\mathcal{C}$,
$a \in P(X)$.

General definition:

A topos $\mathcal{E}$ is a category which is both a quotient and a subcategory of some $\widehat{\mathcal{C}}$ (for $\mathcal{C}$ = small category) in the sense that there are 2 functors

$$(\widehat{\mathcal{C}} \xrightarrow{j^*} \mathcal{E}, \quad \mathcal{E} \xhookrightarrow{j_*} \widehat{\mathcal{C}})$$

such that

- *j^* (= "sheafification functor") is left-adjoint to j_*,*
- *j_* is fully faithful (⇔ $j^* \circ j_* \to \mathrm{id}_{\mathcal{E}}$ is an isomorphism),*
- *j^* respects finite limits.*

Remarks:

(i) j_* respects arbitrary limits.

(ii) j^* respects arbitrary colimits.

(iii) If $\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \mathcal{E}$, any object E of $\mathcal{E}$ can be written

$$E = \varinjlim_{(X,a) \in \int j_* E} \ell(X) \,.$$

Grothendieck topologies:

Observation:

If $X =$ object of $\mathcal{C}$, a subobject of $y(X) = \mathrm{Hom}(\bullet, X)$ in $\widehat{\mathcal{C}}$

$$S \hookrightarrow y(X)$$

is a family S of morphisms $X' \to X$ stable under composition with any morphism $X'' \to X'$. This is called a "sieve".

Definition:

- *For* $\mathcal{E} =$ *topos presented as* $(\widehat{\mathcal{C}} \xrightarrow{j^*} \mathcal{E},\ \mathcal{E} \xhookrightarrow{j_*} \widehat{\mathcal{C}})$ *, a sieve*

  $$S \hookrightarrow y(X)$$

  is called "covering" if

  $$(j^* S \hookrightarrow j^* y(X)) = \textit{isomorphism in } \mathcal{E}.$$

- $J(X) =$ *family of covering sieves of* X.
- $J =$ *indexed family of all* $J(X)$, $X \in \mathrm{Ob}(\mathcal{C})$,
  $=$ *topology on* $\mathcal{C}$.

Theorem:

(i) *If $J =$ topology on $\mathcal{C}$, it verifies:*

(Maximality) For any $X =$ object of $\mathcal{C}$, $y(X) \in J(X)$.

(Stability) For any $X' \xrightarrow{f} X$ and $S \in J(X)$, $f^* S = S \times_{y(X)} y(X') \in J(X')$.

(Transitivity) If $S \in J(X)$ and $S' =$ sieve of X such that $f^* S' \in J(X')$ for any $(X' \xrightarrow{f} X) \in S$, then $S' \in J(X)$.

(ii) *Conversely, any indexed family of sieves*

$$J = \{J(X) \mid X = \text{ object of } \mathcal{C}\}$$

which verifies those three axioms defines a topos $\mathcal{E} = \widehat{\mathcal{C}}_J$

$$(\widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J, \;\; \widehat{\mathcal{C}}_J \xhookrightarrow{j_*} \widehat{\mathcal{C}}\,).$$

Properties of toposes:

$=$ same constructive categorical properties as Set:

- Locally small category.
- Arbitrary colimits and arbitrary limits.
- For any morphism $E \to S$, the pull-back functor
$$E \times_S \bullet$$
respects arbitrary colimits.
- Sums are disjoint.
- For any object E, quotients
$$E \twoheadrightarrow Q$$
correspond to equivalence relations
$$R \hookrightarrow E \times E$$
by $R = E \times_Q E$,
$Q = \varinjlim(R \rightrightarrows E)$.
- For any object E, subobjects of E and quotients of E make up sets.
- If $u : E' \to E$ is a monomorphism and an epimorphism, $u =$ isomorphism.

Theorem (Giraud):

Let

$\mathcal{E}$ = *category verifying these properties,*

$\mathcal{C}$ = *small full subcategory of* $\mathcal{E}$
such that any object X *of* $\mathcal{E}$
has an epimorphic family
$X_i \to X$ *with* $X_i =$ *object of* $\mathcal{C}$, $\forall\, i$,

J = *topology of* $\mathcal{C}$
for which $S \hookrightarrow y(X)$ *is covering*
if it contains an epimorphic family
$$(X_i \to X)\,.$$

Then

$$\mathcal{E} \cong \widehat{\mathcal{C}}_J$$

is a topos.

Basic examples of morphisms of toposes:

- If
$$X \xrightarrow{f} Y$$
is a continuous map between topological spaces, it induces a pair of adjoint functors:
$$\begin{array}{ccc}(\mathcal{E}_Y \xrightarrow{f^*} \mathcal{E}_X\ , & & \mathcal{E}_X \xrightarrow{f_*} \mathcal{E}_Y) \\ \| & & \| \\ \text{pull-back} & & \text{push-forward}\end{array}$$
Furthermore, f^* respects finite limits.
- If $J =$ topology on $\mathcal{C} =$ small category,
$$(\widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J, \ \ \widehat{\mathcal{C}}_J \overset{j_*}{\hookrightarrow} \widehat{\mathcal{C}}\ ).$$
- If $\rho : \mathcal{C} \to \mathcal{D}$ is a functor between two small categories, it induces:
$$(\rho^* : \widehat{\mathcal{D}} \to \widehat{\mathcal{C}}) = \text{composition with } \rho,$$
$$\rho_* = \text{right adjoint of } \rho^*,$$
$$\rho_! = \text{left adjoint of } \rho^*.$$

$\Rightarrow$ Adjoint pair (ρ^*, ρ_*) such that ρ^* respects (arbitrary) limits.

General definition:

(i) *A morphism of toposes*

$$f : \mathcal{E}' \longrightarrow \mathcal{E}$$

is a pair of adjoint functor

$$f = (\mathcal{E} \xrightarrow{f^*} \mathcal{E}', \mathcal{E}' \xrightarrow{f_*} \mathcal{E})$$

whose pull-back component

$$f^* \quad \text{respects finite limits.}$$

(ii) *A transform of morphisms of toposes*

$$(\mathcal{E}' \xrightarrow{f} \mathcal{E}) \longrightarrow (\mathcal{E}' \xrightarrow{g} \mathcal{E})$$

is a transform of functors

$$f^* \longrightarrow g^* \quad \text{(or equivalently} \quad g_* \to f_*\text{).}$$

(iii) *The category of points of a topos $\mathcal{E}$*
is the category of morphisms of toposes

$$\mathrm{Set} \longrightarrow \mathcal{E}\,.$$

(iv) *An embedding of toposes is a morphism*

$$j = (\mathcal{E} \xrightarrow{j^*} \mathcal{E}', \mathcal{E}' \xrightarrow{j_*} \mathcal{E})$$

whose push-forward component j_* is fully faithful.

Remarks:

- Any continuous map between topological spaces

$$f : X \longrightarrow Y$$

induces a morphism of toposes

$$(f^* : \mathcal{E}_Y \longrightarrow \mathcal{E}_X \, , \; f_* : \mathcal{E}_X \longrightarrow \mathcal{E}_Y) \, .$$

This is one-to-one if Y is sober.

- Any topology J on $\mathcal{C}$ defines an embedding

$$\widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}} \, .$$

- Any functor between small categories

$$\rho : \mathcal{C} \longrightarrow \mathcal{D}$$

defines a morphism of toposes

$$(\rho^* : \widehat{\mathcal{D}} \longrightarrow \widehat{\mathcal{C}} , \; \rho_* : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{D}})$$

whose pull-back component ρ^* also has a left adjoint

$$\rho_! : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{D}} \, .$$

Languages of first-order theories

Definition:

A first-order language (or signature) Σ consists in

- *a family of "sorts" (= names of objects)*

$$G, R, K, V, M, \cdots$$

- *a family of "function symbols" (= names of morphisms)*

$$f : A_1 \cdots A_n \longrightarrow B \qquad \textit{(where } A_1, \cdots, A_n, B = \textit{sorts),}$$

- *a family of "relation symbols" (= names of relations)*

$$R \rightarrowtail A_1 \cdots A_n \qquad \textit{(where } A_1, \cdots, A_n = \textit{sorts).}$$

Remark:

If $n = 0$:

$$(f : \rightarrow B) = \text{"constant symbol"},$$
$$(R \rightarrowtail\ ) = \text{"proposition symbol"}.$$

Examples:

(i) Language of the theory of groups:

- one sort G (= "group"),
- three function symbols

$$\begin{array}{rcrcll} \cdot & : & GG & \to & G & \text{(multiplication)},\\ 1 & : & & \to & G & \text{(unit element)},\\ (\bullet)^{-1} & : & G & \to & G & \text{(inverse)}, \end{array}$$

- no relation symbol.

(ii) Language of the theory of equivalence relations

- one sort E (= underlying object),
- no function symbol,
- one relation symbol

$$R \rightarrowtail EE \quad (= \text{equivalence relation}).$$

Interpretations of signatures

Definition:

Let $\Sigma =$ *signature,*

$\mathcal{E} =$ *topos (or category with finite products, including a terminal object* $1_{\mathcal{E}}$*).*

(i) *A* Σ*-structure* M *in* $\mathcal{E}$ *is a map*

$$\text{sort } A \longmapsto MA = \text{object of } \mathcal{E},$$

$$\begin{pmatrix} \text{function symbol} \\ f : A_1 \cdots A_n \to B \end{pmatrix} \longmapsto \begin{pmatrix} \text{morphism} \\ MA_1 \times \cdots \times MA_n \xrightarrow{Mf} MB \end{pmatrix}$$

and, if $n = 0$*,*

$$(c : \to B) \longmapsto (1_{\mathcal{E}} \xrightarrow{Mc} MB)\,,$$

$$\begin{pmatrix} \text{relation symbol} \\ R \rightarrowtail A_1 \cdots A_n \end{pmatrix} \longmapsto \begin{pmatrix} \text{subobject} \\ MR \hookrightarrow MA_1 \times \cdots \times MA_n \end{pmatrix}$$

(ii) *A morphism of Σ-structures in $\mathcal{E}$*

$$u : M \longrightarrow N$$

is a map

$$\text{sort } A \longmapsto \begin{pmatrix} \text{morphism of } \mathcal{E} \\ u_A : MA \to NA \end{pmatrix}$$

such that

- *for any $f : A_1 \cdots A_n \to B$,*

$$\begin{array}{ccc} MA_1 \times \cdots \times MA_n & \xrightarrow{Mf} & MB \\ {\scriptstyle u_{A_1} \times \cdots \times u_{A_n}} \downarrow & & \downarrow {\scriptstyle u_B} \\ NA_1 \times \cdots \times NA_n & \xrightarrow{Nf} & NB \end{array}$$

  is commutative,
- *for any $R \rightarrowtail A_1 \cdots A_n$, there is a factorization:*

$$\begin{array}{ccc} MR & \hookrightarrow & MA_1 \times \cdots \times MA_n \\ \downarrow & & \downarrow {\scriptstyle u_{A_1} \times \cdots \times u_{A_n}} \\ NR & \hookrightarrow & NA_1 \times \cdots \times NA_n \end{array}$$

Consequences:

$\Sigma =$ signature.

(i) If $\mathcal{E} =$ topos or category with finite products,
$\Sigma\text{-str}(\mathcal{E}) =$ category of Σ-structures in $\mathcal{E}$.

(ii) If

$$F : \mathcal{E}' \longrightarrow \mathcal{E}$$

$=$ functor which respects finite limits
(or, more generally, which respects finite products and monomorphisms),
there is an induced functor

$$F : \Sigma\text{-str}(\mathcal{E}') \longrightarrow \Sigma\text{-str}(\mathcal{E}) .$$

(iii) In particular, if

$$f : \mathcal{E}' \longrightarrow \mathcal{E}$$

is a morphism of toposes, it induces adjoint functors

$$f^* : \Sigma\text{-str}(\mathcal{E}) \longrightarrow \Sigma\text{-str}(\mathcal{E}') ,$$

$$f_* : \Sigma\text{-str}(\mathcal{E}') \longrightarrow \Sigma\text{-str}(\mathcal{E}) .$$

The notion of "geometric" (first-order) theory:

Definition:

(i) *A "geometric" theory consists in*

- *a signature* Σ*,*
- *a collection of "sequents"*

$$\varphi \vdash_{\vec{x}} \psi$$

relying "geometric" formulas of Σ

$$\varphi \,,\ \psi$$

in the same family of variables

$\vec{x} = (x_1^{A_1} \cdots x_n^{A_n})$ *(called a "context")*

associated with sorts of Σ

$$A_1 \cdots A_n \,.$$

(ii) *A "geometric" formula is built from "atomic" formulas using symbols*

$\bigwedge =$ *finite conjunction (plus the empty conjunction* $= \top =$ *"true"),*
$\bigvee =$ *arbitrary disjunctions (plus the empty disjunction* $= \bot =$ *"false"),*
$\exists =$ *existential quantifyer in part of the variables.*

(iii) *An "atomic" formula is deduced from*

a relation formula $R(x_1^{A_1} \cdots x_n^{A_n})$

(for a relation symbol $R \rightarrowtail A_1 \cdots A_n$ *of* Σ*)*

or an equality formula

$$(x_1^{A_1} \cdots x_n^{A_n}) = (y_1^{A_1} \cdots y_n^{A_n})$$

by substitutions of some variables by "terms".

(iv) *A term is deduced from an expression* $f_0 = f(x_1^{A_1} \cdots x_n^{A_n})$
(for a function symbol $f : A_1 \cdots A_n \to B$*)*
by an inductive process

$$f_0, f_1, \cdots, f_k$$

where each f_i *is deduced from*

$$f_{i-1} = f_{i-1}(z_1^{C_1} \cdots z_m^{C_m})$$

by replacing some variable $z_i^{C_i}$ *by an expression*

$$g(w_1^{D_1} \cdots w_\ell^{D_\ell})$$

for a function symbol $g : D_1 \cdots D_\ell \to C_i$*.*

Remark:

A (infinitary) first-order theory
allows axioms of the form

$$\varphi \vdash_{\vec{x}} \psi$$

on general "first-order formulas"

$$\varphi \,,\ \psi$$

built from atomic formulas
with the symbols

$\bigwedge$ = arbitrary conjunctions (plus $\top$),

$\bigvee$ = arbitrary disjunctions (plus $\bot$),

$\exists$ = existential quantifyer,

$\forall$ = universal quantifyer,

$\Rightarrow$ = implication,

$\neg$ = negation.

Interpretation of terms:

Definition:

Let $\Sigma =$ *signature,*

$f(x_1^{A_1} \cdots x_n^{A_n}) =$ *term with values in a sort* B*,*

$\mathcal{E} =$ *topos (or a category with finite products)*

$M = \Sigma$*-structure.*

Then $f(x_1^{A_1} \cdots x_n^{A_n})$ *is interpreted in* M *as a morphism*

$$Mf(x_1^{A_1} \cdots x_n^{A_n}) : MA_1 \times \cdots \times MA_n \longrightarrow MB$$

is an inductive way:

- *If* $f = f_0(x_1^{A_1} \cdots x_n^{A_n})$ *is associated with a function symbol*

  $$f_0 : A_1 \cdots A_n \to B\,,$$

  $Mf = Mf_0$*.*
- *If* $f = f_k(x_1^{A_1} \cdots x_n^{A_n})$ *is deduced from*

  $$f_{k-1}(z_1^{C_1} \cdots z_m^{C_m})$$

  by a substitution

  $$z_i^{C_i} = g(w_1^{D_1} \cdots w_\ell^{D_\ell})$$

  (for a function symbol $g : D_1 \cdots D_\ell \to C_i$*),*

  Mf_k *is the composite of* Mf_{k-1} *with the product of* Mg *and* id_{MC_j}*,* $j \neq i$*.*

Interpretation of atomic formulas

Definition:

Let $\Sigma =$ *signature,*

$\mathcal{E} =$ *topos (or a category with finite limits and smallest subobjects),*

$M = \Sigma$*-structure,*

$\varphi = \varphi(x_1^{A_1} \cdots x_n^{A_n}) =$ *atomic formula.*

(i) *If* $\varphi = \top$,

$$M\varphi = MA_1 \times \cdots \times MA_n \text{ is the biggest subobject.}$$

(ii) *If* $\varphi = \bot$,

$$M\varphi = \emptyset_{MA_1 \times \cdots \times MA_n} \quad \text{is the smallest (empty) subobject of } MA_1 \times \cdots \times MA_n.$$

(iii) *If* $\varphi = R(x_1^{A_1} \cdots x_n^{A_n})$ *for a relation symbol* $R \rightarrowtail A_1 \cdots A_n$, $\quad M\varphi = MR.$

(iv) *If* φ *is an equality relation* $(x_1^{A_1} \cdots x_n^{A_n}) = (y_1^{A_1} \cdots y_n^{A_n})$,
$M\varphi$ *is the diagonal subobject*

$$MA_1 \times \cdots \times MA_n \hookrightarrow MA_1 \times \cdots \times MA_n \times MA_1 \times \cdots \times MA_n\,.$$

(v) *If* $\varphi = \varphi_k$ *is deduced from* φ_{k-1} *by a substitution*

$$z_i^{C_i} = g(w_1^{D_1} \cdots w_\ell^{D_\ell})$$

the subobject $M\varphi_k$ *is deduced from the subobject* $M\varphi_{k-1}$
by base change along $Mg : MD_1 \times \cdots \times MD_\ell \to MC_i$.

Interpretation of geometric formulas

Definition:

Let $\Sigma =$ *signature,*
$\mathcal{E} =$ *topos (or a category with enough structures),*
$M = \Sigma$*-structure,*
$\varphi =$ *geometric formula.*

(i) *If* $\varphi = \varphi(\vec{x}) = \varphi_1(\vec{x}) \wedge \cdots \wedge \varphi_k(\vec{x})$
and $\vec{x} = (x_1^{A_1} \cdots x_n^{A_n})$, *the subobject*

$$M\varphi \hookrightarrow MA_1 \times \cdots \times MA_n$$

is the intersection (= fiber product) of the subobjects

$$M\varphi_i \hookrightarrow MA_1 \times \cdots \times MA_n \,.$$

(ii) *If* $\varphi = \varphi(\vec{x}) = \bigvee\limits_{i \in I} \varphi_i(\vec{x})$, *the subobject*

$$M\varphi \hookrightarrow MA_1 \times \cdots \times MA_n$$

is the union

$$\varinjlim \left(\coprod_{i,j} M\varphi_i \times_{MA_1 \times \cdots \times MA_n} M\varphi_j \rightrightarrows \coprod_i M\varphi_i \right)$$

of the subobjects

$$M\varphi_i \hookrightarrow MA_1 \times \cdots \times MA_n \,.$$

(iii) *If* $\varphi = \varphi(\vec{x}) = (\exists\, \vec{y})\, \psi(\vec{x}, \vec{y})$
with $\vec{x} = (x_1^{A_1} \cdots x_n^{A_n})$, $\vec{y} = (y_1^{B_1} \cdots y_m^{B_m})$, *the subobject*

$$M\varphi \hookrightarrow MA_1 \times \cdots \times MA_n$$

is the image

$$\varinjlim\, (M\psi \times_{MA_1 \times \cdots \times MA_n} M\psi \rightrightarrows M\psi)$$

of the composed morphism

$$M\psi \hookrightarrow MA_1 \times \cdots \times MA_n \times MB_1 \times \cdots \times MB_m \to MA_1 \times \cdots \times MA_n\,.$$

Remark:
Interpretations of geometric formulas
only use finite limits and arbitrary colimits.
They are preserved by arbitrary base change.
They are also preserved by pull-back functors

$$f^* : \mathcal{E} \longrightarrow \mathcal{E}'$$

associated with toposes morphisms

$$f : \mathcal{E}' \longrightarrow \mathcal{E}\,.$$

Remarks:

(i) One can prove that arbitrary first-order formulas (which also use symbols $\wedge, \forall, \Rightarrow, \neg$) are interpretable in any topos $\mathcal{E}$.

Moreover, their interpretations are always respected by base change.

(ii) But, for a toposes morphism

$$f : \mathcal{E}' \longrightarrow \mathcal{E} \, ,$$

the functor

$$f^* : \Sigma\text{-str}(\mathcal{E}) \longrightarrow \Sigma\text{-str}(\mathcal{E}')$$

doesn't respect in general the interpretations of these symbols.

Models of geometric first-order theory:

Definition:

Let $\Sigma =$ *signature,*

$\mathbb{T} =$ *(geometric) first-order theory of signature* Σ,

$\mathcal{E} =$ *topos (or a category with enough structures).*

Then:

(i) *A* Σ*-structure in* $\mathcal{E}$

$$M$$

is a "$\mathbb{T}$*-model" if, for any axiom of* $\mathbb{T}$

$$\varphi \vdash_{\vec{x}} \psi \qquad \text{of context } \vec{x} = (x_1^{A_1} \cdots x_n^{A_n}),$$

the subobjects

$$\begin{array}{rcl} M\varphi & \hookrightarrow & MA_1 \times \cdots \times MA_n, \\ M\psi & \hookrightarrow & MA_1 \times \cdots \times MA_n \end{array}$$

verify the inclusion relation

$$M\varphi \leq M\psi .$$

(ii) *A morphism of* $\mathbb{T}$*-models in* $\mathcal{E}$

$$M \longrightarrow N$$

is a morphism of the underlying Σ*-structures.*

Consequences:

(i) For any $\mathcal{E}$, the category of $\mathbb{T}$-models in $\mathcal{E}$

$$\mathbb{T}\text{-mod}(\mathcal{E})$$

is defined as a full subcategory of

$$\Sigma\text{-str}(\mathcal{E})\,.$$

(ii) If $\mathbb{T}$ is geometric, any toposes morphism

$$(f : \mathcal{E}' \to \mathcal{E}) = (f^*, f_*)$$

induces a functor

$$f^* : \mathbb{T}\text{-mod}(\mathcal{E}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{E}')\,.$$

(iii) If $\mathbb{T}$ is geometric and $M = \mathbb{T}$-model in a topos $\mathcal{E}$, there is an induced functor

$$\begin{array}{ccc} (f : \mathcal{E}' \to \mathcal{E}) & \longmapsto & f^* M\,, \\ \left\{ \begin{array}{c} \text{category of} \\ \text{toposes morphisms} \\ \mathcal{E}' \to \mathcal{E} \end{array} \right\} & \longrightarrow & \mathbb{T}\text{-mod}(\mathcal{E}') \\ & & \| \\ & & \text{category of} \\ & & \mathbb{T}\text{-models in } \mathcal{E}'\,. \end{array}$$

(iv) If $(\mathcal{E}_{\mathbb{T}}, U_{\mathbb{T}})$ is such that

$$(f : \mathcal{E} \to \mathcal{E}_{\mathbb{T}}) \longmapsto f^* U_{\mathbb{T}}$$

is an equivalence of categories
for any topos $\mathcal{E}$,
then

$$(\mathcal{E}_{\mathbb{T}}, U_{\mathbb{T}})$$

is uniquely determined
up to equivalence.

Lecture II:

Syntactic categories and classifying toposes

What we want to do:

Start from $\mathbb{T}$ = "geometric" first-order theory.

Construct

$$\begin{cases} \bullet \quad \text{a category } \mathcal{C}_{\mathbb{T}} \text{ with enough properties for } \mathbb{T}\text{-mod}\,(\mathcal{C}_{\mathbb{T}}) \text{ to be defined,} \\ \bullet \quad \text{a model } M_{\mathbb{T}} \text{ of } \mathbb{T} \text{ in } \mathcal{C}_{\mathbb{T}}, \\ \bullet \quad \text{a topology } J_{\mathbb{T}} \text{ on } \mathcal{C}_{\mathbb{T}} \end{cases}$$

such that, denoting

$$\begin{cases} \bullet \quad \mathcal{E}_{\mathbb{T}} = \text{quotient topos of } \widehat{\mathcal{C}}_{\mathbb{T}} \text{ by } J_{\mathbb{T}}, \\ \bullet \quad U_{\mathbb{T}} = \mathbb{T}\text{-model in } \mathcal{E}_{\mathbb{T}} \text{ image of } M_{\mathbb{T}} \text{ by} \\ \qquad\qquad \ell : \mathcal{C}_{\mathbb{T}} \xrightarrow{\ y\ } \widehat{\mathcal{C}}_{\mathbb{T}} \xrightarrow{\ j^*\ } \mathcal{E}_{\mathbb{T}}\,, \end{cases}$$

the functor

$$\begin{array}{ccc} (f : \mathcal{E} \to \mathcal{E}_{\mathbb{T}}) & \longmapsto & f^* U_{\mathbb{T}}\,, \\ \left\{ \begin{matrix} \text{toposes morphisms} \\ \mathcal{E} \to \mathcal{E}_{\mathbb{T}} \end{matrix} \right\} & \longrightarrow & \mathbb{T}\text{-mod}(\mathcal{E}) \end{array}$$

is an equivalence for any topos $\mathcal{E}$.

Geometric categories:

We want $\mathcal{C}_{\mathbb{T}}$ to be "geometric" in the following sense:

Definition:

(i) *A (locally small) category $\mathcal{C}$ is geometric if*

- *it has finite limits,*
- *in particular, any morphism*

$$p : X \longrightarrow Y$$

defines a functor on categories of subobjects

$$\begin{array}{rccc} p^* : & \Omega(Y) & \longrightarrow & \Omega(X)\,, \\ & (S \hookrightarrow Y) & \longmapsto & (S \times_Y X \hookrightarrow X)\,, \end{array}$$

- *this functor has a left adjoint*

$$\exists_p : \Omega(X) \longrightarrow \Omega(Y)$$

and it commutes with base change:

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ {\scriptstyle p'}\downarrow & \square & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{y} & Y \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \Omega(X) & \xrightarrow{x^*} & \Omega(X') \\ {\scriptstyle \exists_p}\downarrow & & \downarrow{\scriptstyle \exists_{p'}} \\ \Omega(Y) & \xrightarrow{y^*} & \Omega(Y') \end{array}$$

- *any family of subobjects*

$$S_i \hookrightarrow Y\,, \quad i \in I\,,$$

has a union

$$\bigvee_{i \in I} S_i \hookrightarrow Y$$

such that, for any $S \hookrightarrow Y$,

$$\bigvee_{i \in I} S_i \leq S \qquad \text{iff} \qquad S_i \leq S\,, \quad \forall\, i\,,$$

and it commutes with base change:

$$(X \xrightarrow{p} Y) \Longrightarrow p^* \bigvee_{i \in I} S_i = \bigvee_{i \in I} p^* S_i$$

(ii) *A functor between geometric categories*

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

is "geometric" if

- *it respects finite limits,*
- *it respects functors $\exists_p$ in the sense that*

$$\begin{pmatrix} S & \hookrightarrow & X \\ & & \downarrow p \\ & & Y \end{pmatrix} \Longrightarrow F(\exists_p S) = \exists_{F(p)} F(S)\,,$$

- *it respects arbitrary unions $\bigvee\limits_{i \in I}$.*

Remarks:

(i) For $\Sigma =$ signature,

$\mathcal{C} =$ geometric category,

$M = \Sigma$-structure in $\mathcal{C}$,

any geometric formula

$$\varphi = \varphi(x_1^{A_1} \cdots x_n^{A_n}) = \varphi(\vec{x})$$

is interpretable as a subobject

$$M\varphi(\vec{x}) \hookrightarrow MA_1 \times \cdots \times MA_n \,.$$

(ii) If $\mathbb{T} =$ geometric theory,

M is a $\mathbb{T}$-model in $\mathcal{C}$

if and only if, for any axiom of $\mathbb{T}$

$$\varphi \vdash_{\vec{x}} \psi \,,$$

M verifies the inclusion relation

$$M\varphi(\vec{x}) \leq M\psi(\vec{x}) \,.$$

There is an induced full subcategory

$$\mathbb{T}\text{-mod}(\mathcal{C}) \hookrightarrow \Sigma\text{-str}(\mathcal{C}) \,.$$

(iii) Any geometric functor

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

induces a functor

$$F : \mathbb{T}\text{-mod}(\mathcal{C}) \longrightarrow \mathbb{T}\text{-mod}(\mathcal{D}) \,.$$

Characterization of the syntactic category:

Theorem:

Let $\mathbb{T} =$ *geometric theory.*

(i) *There exists a (essentially small) geometric category*

$$\mathcal{C}_{\mathbb{T}}$$

endowed with a $\mathbb{T}$*-model*

$$M_{\mathbb{T}}\,,$$

such that, for any geometric category $\mathcal{C}$*, the functor*

$$\begin{Bmatrix} \textit{geometric functors} \\ F : \mathcal{C}_{\mathbb{T}} \to \mathcal{C} \end{Bmatrix} \longrightarrow \mathbb{T}\text{-}\mathit{mod}(\mathcal{C})\,,$$
$$F \longmapsto F(M_{\mathbb{T}})$$

is an equivalence.

(ii) *The couple*

$$(\mathcal{C}_{\mathbb{T}}, M_{\mathbb{T}})$$

is well defined up to equivalence.

Construction of the syntactic category:

Definition:
Let $\mathbb{T}$ = geometric theory of signature Σ.

(i) *Objects of $\mathcal{C}_{\mathbb{T}}$ are geometric formulas of Σ*

$$\varphi(\vec{x}) = \varphi(x_1^{A_1} \cdots x_n^{A_n}),$$

considered up to substitution of variables

$$(x_1^{A_1} \cdots x_n^{A_n}) \leftrightarrow (y_1^{A_1} \cdots y_n^{A_n}).$$

(ii) *Morphisms of $\mathcal{C}_{\mathbb{T}}$*

$$\varphi(\vec{x}) \longrightarrow \psi(\vec{y})$$

(if $\vec{x} = (x_1^{A_1} \cdots x_n^{A_n})$ and $\vec{y} = (y_1^{B_1} \cdots y_m^{B_m})$ are disjoint) are geometric formulas

$$\theta(\vec{x}, \vec{y})$$

that are "provably functional" in the sense that the sequents

$$\begin{cases} \theta & \vdash_{\vec{x},\vec{y}} & \varphi \wedge \psi\,, \\ \varphi & \vdash_{\vec{x}} & (\exists\, \vec{y})\, \theta(\vec{x}, \vec{y})\,, \\ \theta(\vec{x}, \vec{y}) \wedge \theta(\vec{x}, \vec{y}\,') & \vdash_{\vec{x},\vec{y},\vec{y}\,'} & \vec{y} = \vec{y}\,' \end{cases}$$

are provable in $\mathbb{T}$,
up to $\mathbb{T}$-provable equivalence of these formulas.

(iii) *The composite of two morphisms*

$$\begin{array}{lcccl} \theta(\vec{x},\vec{y}) & : & \varphi(\vec{x}) & \longrightarrow & \psi(\vec{y})\,, \\ \theta'(\vec{y},\vec{z}) & : & \psi(\vec{y}) & \longrightarrow & \chi(\vec{z}) \end{array}$$

is defined as the provably functional geometric formula

$$(\exists\,\vec{y})(\theta(\vec{x},\vec{y}) \wedge \theta'(\vec{y},\vec{z})) : \varphi(\vec{x}) \longrightarrow \chi(\vec{z})\,.$$

Remarks:

(i) Objects of $\mathcal{C}_{\mathbb{T}}$ only depend on the signature Σ of $\mathbb{T}$.

(ii) Morphisms of $\mathcal{C}_{\mathbb{T}}$ depend on the notion of

"$\mathbb{T}$-provability".

Definition:
A sequent between geometric formulas of a signature Σ

$$\varphi \vdash_{\vec{x}} \psi$$

is called "provable" in a theory $\mathbb{T}$ *of signature* Σ
if it can be deduced from the axioms of $\mathbb{T}$

$$\varphi_i \vdash_{\vec{x}_i} \psi_i$$

by a combination of the following rules:

(1) Cut rule:

If $\varphi_1 \vdash_{\vec{x}} \varphi_2$ *and* $\varphi_2 \vdash_{\vec{x}} \varphi_3$, *then* $\varphi_1 \vdash_{\vec{x}} \varphi_3$.

(2) Identity rule:

$\top \vdash_{\vec{x}} (f = f)$ *for any term* f.

(3) Equality rules:

- *If* $\top \vdash_{\vec{x}} f_1 = f_2$, *then* $\top \vdash_{\vec{x}} f_2 = f_1$.
- *If* $\top \vdash_{\vec{x}} f_1 = f_2$ *and* $\top \vdash_{\vec{x}} f_2 = f_3$, *then* $\top \vdash_{\vec{x}} f_1 = f_3$.

(4) Substitution rules:

- *If* f_1, f_2, f *are terms and* f_1', f_2' *are deduced from* f_1, f_2 *by substitution of* f *to some variable, [resp. of* f_1, f_2 *to some variable of* f*],*

  then $\top \vdash f_1 = f_2$

  implies $\top \vdash f_1' = f_2'$.

- *If* $f_1, f_2 =$ *terms,*
  $R =$ *relation,*
  R_1, R_2 *deduced from* R
  by substitution of f_1, f_2 *to some variable,*

  then $\top \vdash f_1 = f_2$

  implies $R_1 \vdash R_2$ *and* $R_2 \vdash R_1$.

(5) Rules of finitary conjunctions:

- $\varphi \vdash \top$ *holds for any* φ *in a context* $\vec{x}$.
- *For any* $\varphi, \varphi_1, \cdots, \varphi_k$ *in a context* $\vec{x}$,

$$\varphi \vdash \varphi_1 \wedge \cdots \wedge \varphi_k$$

is equivalent to

$$\varphi \vdash \varphi_i \quad \text{for any } i, \quad 1 \leq i \leq k.$$

(6) Rules of infinitary disjunctions:

- $\bot \vdash \varphi$ *holds for any* φ *in a context* $\vec{x}$.
- *For any* φ *and* φ_i, $i \in I$, *in a context* $\vec{x}$,

$$\bigvee_{i \in I} \varphi_i \vdash \varphi$$

is equivalent to

$$\varphi_i \vdash \varphi \quad \text{for any} \quad i \in I.$$

(7) Distributivity rules:

- *For any* φ *and* φ_i*,* $i \in I$*, in a context* $\vec{x}$*,*

$$\varphi \wedge \left(\bigvee_{i \in I} \varphi_i \right) \vdash_{\vec{x}} \bigvee_{i \in I} (\varphi \wedge \varphi_i)$$

always holds (as well as the converse sequent).

(8) Rules of existential quantification:

- *For any* φ *in a context* $(\vec{x}, \vec{y})$
and any ψ *in the context* $\vec{x}$*,*

$$\varphi \vdash_{\vec{x}, \vec{y}} \psi$$

is equivalent to

$$(\exists \, \vec{y}) \, \varphi \vdash_{\vec{x}} \psi \, .$$

(9) Frobenius rule:

- *For any* φ *in a context* $(\vec{x}, \vec{y})$
and any ψ *in the context* $\vec{x}$

$$(\exists \, \vec{y}) \, \varphi \wedge \psi \vdash_{\vec{x}} (\exists \, \vec{y})(\varphi \wedge \psi)$$

always holds (as well as the converse sequent).

Quotient theories:

Definition:
$\Sigma =$ *signature,*
$\mathbb{T}, \mathbb{T}' =$ *geometric theories in the signature* Σ*.*

(i) $\mathbb{T}'$ *is called a "quotient" of* $\mathbb{T}$
if any geometric sequent of Σ

$$\varphi \vdash_{\vec{x}} \psi$$

which is provable in $\mathbb{T}$
is also provable in $\mathbb{T}'$*.*

(ii) $\mathbb{T}$ *and* $\mathbb{T}'$ *are called "equivalent" if*
$\mathbb{T}$*-provable* $\Leftrightarrow$ $\mathbb{T}'$*-provable.*

Remarks:

(i) If $\mathbb{T}'$ is a quotient of $\mathbb{T}$,
$\mathcal{C}_{\mathbb{T}}$ as a natural functor to $\mathcal{C}_{\mathbb{T}'}$ with the same objects.

(ii) If $\mathbb{T}$, $\mathbb{T}'$ are equivalent,

$$\mathcal{C}_{\mathbb{T}} = \mathcal{C}_{\mathbb{T}'} \, .$$

Subobjects in syntactic categories:

Proposition:

Let $\mathbb{T} =$ *geometric theory of signature* Σ,
$\mathcal{C}_{\mathbb{T}} =$ *syntactic category of* $\mathbb{T}$,
$\varphi(\vec{x}) = \varphi(x_1^{A_1} \cdots x_n^{A_n}) =$ *object of* $\mathcal{C}_{\mathbb{T}}$
$=$ *geometric formula of* Σ.

Then:

(i) *Subobjects of* $\varphi(\vec{x})$ *in* $\mathcal{C}_{\mathbb{T}}$ *correspond to geometric formulas*

$$\varphi_1(\vec{x})$$

such that the sequent

$$\varphi_1 \vdash_{\vec{x}} \varphi$$

is $\mathbb{T}$*-provable.*

(ii) *Two subobjects of* $\varphi(\vec{x})$

$$\varphi_1(\vec{x}) \qquad \textit{and} \qquad \varphi_2(\vec{x})$$

verify the inclusion relation

$$\varphi_1(\vec{x}) \leq \varphi_2(\vec{x})$$

if and only if the sequent

$$\varphi_1 \vdash_{\vec{x}} \varphi_2$$

is $\mathbb{T}$*-provable.*

The universal model $M_{\mathbb{T}}$ in $\mathcal{C}_{\mathbb{T}}$:

Definition:

Let $\mathbb{T} =$ *geometric theory of signature* Σ,

$\mathcal{C}_{\mathbb{T}} =$ *syntactic category of* $\mathbb{T}$.

Then the Σ*-structure* $M_{\mathbb{T}}$ *in* $\mathcal{C}_{\mathbb{T}}$ *is defined in the following way:*

(i) *For any sort* A, $M_{\mathbb{T}}A$ *is the object of* $\mathcal{C}_{\mathbb{T}}$

$$\top(x^A)\,.$$

(ii) *For any function symbol of* Σ

$$f : A_1 \cdots A_n \longrightarrow B\,,$$

$M_{\mathbb{T}}f$ *is the morphism of* $\mathcal{C}_{\mathbb{T}}$

$$\top(x_1^{A_1} \cdots x_n^{A_n}) \xrightarrow{x^B = f(x_1^{A_1} \cdots x_n^{A_n})} \top(x^B)\,.$$

(iii) *For any relation symbol of* Σ

$$R \rightarrowtail A_1 \cdots A_n\,,$$

$M_{\mathbb{T}}R$ *is the subobject*

$$R(x_1^{A_1} \cdots x_n^{A_n}) \hookrightarrow \top(x_1^{A_1} \cdots x_n^{A_n})\,.$$

Lemma:
For any geometric formula in the signature Σ

$$\varphi(\vec{x}) = \varphi(x_1^{A_1} \cdots x_n^{A_n}),$$

its interpretation in the Σ*-structure* $M_{\mathbb{T}}$ *of* $\mathcal{C}_{\mathbb{T}}$

$$M_{\mathbb{T}}\, \varphi(\vec{x})$$

is the subobject

$$\begin{aligned} \varphi(\vec{x}) \hookrightarrow \top(\vec{x}) &= \top(x_1^{A_1}) \times \cdots \times \top(x_n^{A_n}) \\ &= M_{\mathbb{T}}A_1 \times \cdots \times M_{\mathbb{T}}A_n\,. \end{aligned}$$

Corollary:
A geometric sequent of the signature Σ

$$\varphi \vdash_{\vec{x}} \psi$$

is $\mathbb{T}$*-provable if and only if, as subobjects of* $\top(\vec{x})$*,*

$$\varphi(\vec{x}) \leq \psi(\vec{x}),$$

i.e. if and only if it is verified by $M_{\mathbb{T}}$*.*

In particular, $M_{\mathbb{T}}$ *is a model of* $\mathbb{T}$ *in* $\mathcal{C}_{\mathbb{T}}$*.*

Theorem:

Let $\mathbb{T}$ = *geometric category,*
$\mathcal{C}_{\mathbb{T}}$ = *syntactic category of* $\mathbb{T}$.

Then, for $\mathcal{C}$ = *geometric category, the functor*

$$\begin{array}{ccc} \left\{\begin{array}{c}\textit{geometric functors} \\ \mathcal{C}_{\mathbb{T}} \to \mathcal{C}\end{array}\right\} & \longrightarrow & \mathbb{T}\text{-mod}(\mathcal{C}), \\ (F : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}) & \longmapsto & F(M_{\mathbb{T}}) \end{array}$$

is an equivalence, and a reverse equivalence is

$$\begin{array}{ccc} \mathbb{T}\text{-mod}(\mathcal{C}) & \longrightarrow & \{\textit{geometric functors}\ \mathcal{C}_{\mathbb{T}} \to \mathcal{C}\}, \\ M & \longmapsto & (F_M : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}) \end{array}$$

defined as:

- *for any object* $\varphi(\vec{x})$ *of* $\mathcal{C}_{\mathbb{T}}$,
$$F_M(\varphi(\vec{x})) = M\varphi(\vec{x}),$$
- *for any morphism of* $\mathcal{C}_{\mathbb{T}}$
$$\theta : \varphi(\vec{x}) \xrightarrow{\theta(\vec{x},\vec{y})} \psi(\vec{y}),$$

  $F_M(\theta)$ *is the morphism* $M\varphi(\vec{x}) \to M\psi(\vec{y})$ *whose graph is* $M\theta(\vec{x},\vec{y})$.

The syntactic topology $J_{\mathbb{T}}$ on $\mathcal{C}_{\mathbb{T}}$:

Definition:
A sieve on an object

$$\psi(\vec{y}) \qquad \text{of} \qquad \mathcal{C}_{\mathbb{T}}$$

is called "$J_{\mathbb{T}}$-covering" if it contains a family of morphisms

$$\theta_i(\vec{x}_i, \vec{y}) : \varphi_i(\vec{x}_i) \longrightarrow \psi(\vec{y}), \quad i \in I,$$

whose union of images is the full object, equivalently, such that the sequent

$$\psi(\vec{y}) \vdash_{\vec{y}} \bigvee_{i \in I} (\exists\, \vec{x}_i)\, \theta_i(\vec{x}_i, \vec{y})$$

is $\mathbb{T}$-provable.

Remarks:

(i) $J_{\mathbb{T}}$ is a topology because
the condition for the union of images to be the full object
is preserved by base change in $\mathcal{C}_{\mathbb{T}}$.

(ii) $J_{\mathbb{T}}$ is defined by the categorical structure of $\mathcal{C}_{\mathbb{T}}$.

Diaconescu's equivalence:

Theorem:

Let $\mathcal{C}$ = *(essentially) small category with finite limits,*
J = *topology on* $\mathcal{C}$,
$(\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J)$ = *canonical functor,*
$\mathcal{E}$ = *topos.*

Then:

(i) *For any toposes morphism*

$$f = (f^*, f_*) : \mathcal{E} \longrightarrow \widehat{\mathcal{C}}_J \,,$$

$(f^* \circ \ell : \mathcal{C} \to \mathcal{E})$ *is*

- *"flat" in the sense that it respects finite limits,*
- *"J-continuous" in the sense that it transforms* J*-covering families of* $\mathcal{C}$ *into globally epimorphic families of* $\mathcal{E}$.

(ii) *The functor*

$$\left\{ \begin{array}{c} \text{category of} \\ \text{toposes morphisms} \\ \mathcal{E} \to \widehat{\mathcal{C}}_J \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{category of} \\ \text{flat } J\text{-continuous functors} \\ \mathcal{C} \to \mathcal{E} \end{array} \right\}$$

is an equivalence

Lemma:

Let

$\mathbb{T} =$ *geometric theory,*

$\mathcal{C}_{\mathbb{T}} =$ *associated syntactic category,*

$\mathcal{E} =$ *topos.*

Then a functor

$$F : \mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{E}$$

is "geometric" if and only if it is

- *flat,*
- $J_{\mathbb{T}}$*-continuous,*

  i.e. transforms
  families of morphisms whose union of images is the full object
  into globally epimorphic families of $\mathcal{E}$*.*

Corollary:

Let

$\mathbb{T}$ = *geometric theory,*

$\mathcal{E}_{\mathbb{T}}$ = *quotient topos of* $\widehat{\mathcal{C}}_{\mathbb{T}}$ *by* $J_{\mathbb{T}}$,

$\mathcal{E}$ = *topos.*

Then the composite functor

$$\left\{\begin{array}{c}\textit{category of}\\ \textit{toposes morphisms}\\ \mathcal{E}\to\mathcal{E}_{\mathbb{T}}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\textit{flat } J\textit{-continuous}\\ \textit{functors}\\ \mathcal{C}_{\mathbb{T}}\to\mathcal{E}\end{array}\right\}$$

$$\Vert$$

$$\left\{\begin{array}{c}\textit{geometric functors}\\ \mathcal{C}_{\mathbb{T}}\to\mathcal{E}\end{array}\right\} \longrightarrow \mathbb{T}\text{-mod}(\mathcal{E})$$

is an equivalence of categories.

Definition:

Let

$\mathcal{C}$ = *(essentially) small category,*

J = *topology on* $\mathcal{C}$,

$(\ell : \mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \xrightarrow{j^*} \widehat{\mathcal{C}}_J)$ = *canonical functor.*

Then J is called "subcanonical"
if and only if it verifies the following equivalent conditions:

(1) *The functor*

$$\ell : \mathcal{C} \longrightarrow \widehat{\mathcal{C}}_J$$

is fully faithful.

(2) *The functor*

$$y : \mathcal{C} \longrightarrow \widehat{\mathcal{C}}$$

factorises through $j_* : \widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}$.

Lemma: *Let*

$\mathbb{T}$ = *geometric category,*

$\mathcal{C}_{\mathbb{T}}$ = *syntactic category of* $\mathbb{T}$,

$\mathcal{E}_{\mathbb{T}} = \widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}}$ = *classifying topos of* $\mathbb{T}$,

$(\ell : \mathcal{C}_{\mathbb{T}} \to \mathcal{E}_{\mathbb{T}})$ = *canonical functor.*

Then ℓ *is fully faithful i.e.* $J_{\mathbb{T}}$ *is subcanonical.*

Corollary:
A geometric sequent of $\mathbb{T}$

$$\varphi \vdash_{\vec{x}} \psi$$

is $\mathbb{T}$*-provable if and only if*
it is verified by the universal model of $\mathbb{T}$

$$U_{\mathbb{T}} = \ell(M_{\mathbb{T}}) \qquad \textit{in} \qquad \mathcal{E}_{\mathbb{T}}\,.$$

Consequence:

$$\left(\begin{array}{c}\text{Gödel's}\\ \text{completeness}\\ \text{theorem}\end{array}\right) = \left(\begin{array}{c}\text{Deligne's theorem}\\ \text{on ``coherent'' toposes}\\ \text{having enough points}\end{array}\right).$$

Proposition:

For any topos $\mathcal{E}$,
there are (infinitely many) geometric theories $\mathbb{T}$ such that

$$\mathcal{E}_{\mathbb{T}} \cong \mathcal{E} \, .$$

Hint:

Write $\quad \mathcal{E} \cong \widehat{\mathcal{C}}_J$
for $\quad \mathcal{C} =$ small category with finite limits,
$\quad J =$ topology on $\mathcal{C}$.

Then define $\mathbb{T}$ as the "geometric theory"
of "flat J-continuous" functors on $\mathcal{C}$.

Definition:

Geometric theories

$$\mathbb{T}_1 \qquad \textit{and} \qquad \mathbb{T}_2$$

are called "Morita equivalent" or "semantically equivalent" if

$$\mathcal{E}_{\mathbb{T}_1} \cong \mathcal{E}_{\mathbb{T}_2} \, .$$

Remarks:

(i) The condition

$$\mathcal{E}_{\mathbb{T}_1} \cong \mathcal{E}_{\mathbb{T}_2}$$

is verified if $\mathbb{T}_1$ and $\mathbb{T}_2$ are "syntactically equivalent"
in the sense that their syntactic categories are equivalent

$$\mathcal{C}_{\mathbb{T}_1} \cong \mathcal{C}_{\mathbb{T}_2} \, .$$

(ii) The converse is not true.

Lecture III:

Provability and Grothendieck topologies, presheaf-type theories

The notion of subtopos:

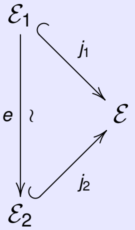
Definition:

(i) *A morphism of toposes*

$$j = (j^*, j_*) : \mathcal{E}' \longrightarrow \mathcal{E}$$

is called an "embedding"
if j_* is fully faithful or, equivalently,
$j^* \circ j_* \to \mathrm{id}_{\mathcal{E}'}$ is an isomorphism.

(ii) *A subtopos of a topos $\mathcal{E}$*
is an equivalence class of embeddings
for the relation defined by diagrams



with $j_2 \circ e \cong j_1$.

Subtoposes and topologies:

If $\mathcal{E} = \widehat{\mathcal{C}}_J =$ topos associated to $\begin{cases} \mathcal{C} = \text{category}, \\ J = \text{topology on } \mathcal{C}, \end{cases}$

any topology $J' \supseteq J$ on $\mathcal{C}$ defines a subtopos

$$\widehat{\mathcal{C}}_{J'} \hookrightarrow \widehat{\mathcal{C}}_J \hookrightarrow \widehat{\mathcal{C}}\,.$$

Conversely:

Proposition:

If $\mathcal{E} = \widehat{\mathcal{C}}_J$, the map

$$\begin{array}{ccc} (J' \supseteq J) & \longmapsto & (\widehat{\mathcal{C}}_{J'} \hookrightarrow \widehat{\mathcal{C}}_J = \mathcal{E}) \\ \left\{ \begin{matrix} \textit{topologies} \\ J' \supseteq J \textit{ on } \mathcal{C} \end{matrix} \right\} & \longmapsto & \left\{ \begin{matrix} \textit{subtoposes} \\ \mathcal{E}' \hookrightarrow \mathcal{E} \end{matrix} \right\} \end{array}$$

is one-to-one.

Remark:
This implies that, for any topos $\mathcal{E}$,
subtoposes of $\mathcal{E}$ make up a set ordered by inclusion.

Subtoposes and quotient theories:

Theorem (Caramello):

Let $\mathbb{T}$ = *geometric theory of signature* Σ.

(i) *For any* $\mathbb{T}'$ = *quotient theory of* $\mathbb{T}$
(= geometric theory of same signature Σ *with more provable sequents),*
$\mathcal{E}_{\mathbb{T}'}$ *is a subtopos of* $\mathcal{E}_{\mathbb{T}}$.

(ii) *The map*

$$\left\{\begin{array}{c}\text{quotient theories } \mathbb{T}' \text{ of } \mathbb{T},\\ \text{considered up to equivalence}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{subtoposes}\\ \mathcal{E}' \hookrightarrow \mathcal{E}_{\mathbb{T}}\end{array}\right\},$$

$$\mathbb{T}' \longmapsto (\mathcal{E}_{\mathbb{T}'} \hookrightarrow \mathcal{E}_{\mathbb{T}})$$

is one-to-one.

Sketch of the proof: One has to prove a correspondence:

$$\left\{\begin{matrix}\text{quotients}\\ \mathbb{T}' \text{ of } \mathbb{T}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{topologies}\\ J \supseteq J_{\mathbb{T}}\\ \text{on } \mathcal{C}_{\mathbb{T}}\end{matrix}\right\}$$

• From a quotient $\mathbb{T}'$ of $\mathbb{T}$ to a topology J on $\mathcal{C}_{\mathbb{T}}$:

For any axiom of $\mathbb{T}'$,

$$\varphi \vdash_{\vec{x}} \psi$$

consider the associated monomorphism of $\mathcal{C}_{\mathbb{T}}$

$$\varphi \wedge \psi(\vec{x}) \hookrightarrow \varphi(\vec{x})$$

and decide it is J-covering.
Define J as the smallest topology on $\mathcal{C}_{\mathbb{T}}$ containing $J_{\mathbb{T}}$ and these coverings.

• From a topology $J \supseteq J_{\mathbb{T}}$ on $\mathcal{C}_{\mathbb{T}}$ to a quotient $\mathbb{T}'$ of $\mathbb{T}$:

For any J-covering family of morphisms of $\mathcal{C}_{\mathbb{T}}$

$$\theta_i(\vec{x}_i, \vec{x}) : \varphi_i(\vec{x}_i) \longrightarrow \varphi(\vec{x})\,, \quad i \in I\,,$$

decide that

$$\varphi(\vec{x}) \vdash_{\vec{x}} \bigvee_{i \in I} (\exists\, \vec{x}_i)\, \theta_i(\vec{x}_i, \vec{x})$$

is an axiom of $\mathbb{T}'$. Define $\mathbb{T}'$ by the collection of these axioms.

Consequence: provability and Grothendieck topologies

Corollary:

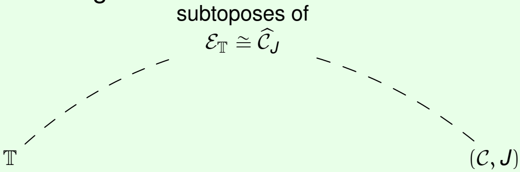
Let $\mathbb{T}$ = *geometric theory,*

$\mathcal{E}_{\mathbb{T}}$ = *classifying topos of* $\mathbb{T}$ *written as* $\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}_J$.

Then there is a natural (constructible) one-to-one correspondence:

$$\left\{ \begin{array}{c} \textit{quotient } \mathbb{T}' \textit{ of } \mathbb{T}, \\ \textit{up to equivalence} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{topologies} \\ J' \supseteq J \textit{ on } \mathcal{C} \end{array} \right\}$$

Remark: This is the bridge:



Remark:

So, a quotient theory $\mathbb{T}'$ of $\mathbb{T}$ is contradictory
if and only if J' is maximal
(i.e. the empty sieve is a J'-covering of any object of $\mathcal{C}$).

Topological translation of provability:

Let

$\mathbb{T}$ = geometric theory of signature Σ,

$(\varphi \vdash_{\vec{x}} \psi)$ = geometric sequent of Σ,

$\mathbb{T}'$ = quotient theory of $\mathbb{T}$ defined by adding the axiom $\varphi \vdash_{\vec{x}} \psi$.

$\mathcal{E}_{\mathbb{T}}$ = classifying topos of $\mathbb{T}$ written as $\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}_J$,

$(J' \supseteq J)$ = topology on $\mathcal{C}$ such that $\widehat{\mathcal{C}}_{J'} = \mathcal{E}_{\mathbb{T}'}$.

Then we have equivalences:

$$(\varphi \vdash_{\vec{x}} \psi) \text{ is provable in } \mathbb{T}$$

$$\Updownarrow$$

$$\mathbb{T}' \text{ is equivalent to } \mathbb{T}$$

$$\Updownarrow$$

$$J' = J\,.$$

How to compute the classifying topos of a theory?

Given a geometric theory $\mathbb{T}$ of signature Σ,

how to construct $\begin{cases} \text{a (essentially) small category } \mathcal{C}, \\ \text{a topology } J \text{ on } \mathcal{C}, \end{cases}$

such that

$$\mathcal{E} \cong \widehat{\mathcal{C}}_J \text{ ?}$$

Remarks:

(i) For any $\mathcal{E}$,
the collection of equivalences

$$\mathcal{E} \cong \widehat{\mathcal{C}}_J$$

is so big that it is not even a set $\cdots$

(ii) One can always take

$$\begin{cases} \mathcal{C} = \mathcal{C}_{\mathbb{T}} = \text{ syntactic category,} \\ J = J_{\mathbb{T}} = \text{ syntactic topology,} \end{cases}$$

but, in general, it cannot be made fully explicit.

Presheaf-type theories:

Definition:
A geometric theory $\mathbb{T}$
is called "of presheaf type"
if its classifying topos $\mathcal{E}_{\mathbb{T}}$
can be written as a presheaf topos

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}$$

for some $\mathcal{C}$ = *(essentially) small category.*

Examples:

(i) If $\mathbb{T}$ = empty theory (no axiom)
on some signature Σ,
$\mathbb{T}$ is of presheaf-type.

(ii) $\mathbb{T}$ is of presheaf-type if it is algebraic, i.e.

$$\begin{cases} \text{no relation symbol,} \\ \text{all axioms have the form } \top \vdash_{\vec{x}} f(\vec{x}) = g(\vec{x}) \text{ for } f, g = \text{terms.} \end{cases}$$

Presheaf toposes:

Theorem:

Let $\mathcal{E} = \widehat{\mathcal{C}}$ for $\mathcal{C} =$ (essentially) small category.

Then:

(i) *The category of points of $\widehat{\mathcal{C}}$ is equivalent to the category of functors*

$$P : \mathcal{C} \longrightarrow \mathrm{Set}$$

which verify the following equivalent properties:

(1) *The category of "elements" of P*

$$\int P = \{(X,x) \mid X = \text{ object of } X,\ x \in P(X)\}$$

is filtering.

(2) *P can be written as a filtering colimit of representable functors $\mathrm{Hom}(X,\bullet)$, $X =$ object of $\mathcal{C}$.*

(ii) *As a consequence,*

$$\mathrm{pt}(\widehat{\mathcal{C}}) \cong \mathrm{Ind}(\mathcal{C}^{\mathrm{op}})\,.$$

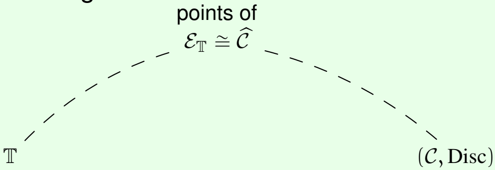
Application to models of presheaf-type theories

Corollary:

If $\mathbb{T}$ = *presheaf-type theory with* $\mathcal{E}_{\mathbb{T}} \simeq \widehat{\mathcal{C}}$,

$\mathbb{T}\text{-mod}(\text{Set}) \simeq \text{Ind}(\mathcal{C}^{\text{op}})$.

Remark: This is the bridge:



Remark:

So, the category

$$\text{Ind}(\mathcal{C}^{\text{op}})$$

is determined by $\mathbb{T}$, up to equivalence.

$\longrightarrow$ Question: To which extent is it possible to recover $\mathcal{C}$?

Finitely presentable models:

Definition:

Let $\mathbb{T}$ = *presheaf-type theory.*

A set-based model M of $\mathbb{T}$ *(*$= $ *object of* $\mathbb{T}\text{-mod}(\mathrm{Set})$*)*

is called "finitely presentable"

if the functor

$$\mathrm{Hom}(M, \bullet) : \mathbb{T}\text{-mod}(\mathrm{Set}) \longrightarrow \mathrm{Set}$$

respects filtering colimits.

Remark:

One can introduce

$$\mathbb{T}\text{-mod}(\mathrm{Set})_{fp}$$

$=$ full subcategory of $\mathbb{T}\text{-mod}(\mathrm{Set})$

on finitely presentable models.

Proposition:

If $\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}$
and so $\mathbb{T}\text{-mod}(\text{Set}) \cong \text{Ind}(\widehat{\mathcal{C}}^{\text{op}})$,
an object M *of* $\text{Ind}(\widehat{\mathcal{C}}^{\text{op}})$
is finitely presentable if and only if there exist

$$\begin{cases} \text{an object } X \text{ of } \mathcal{C}^{\text{op}}, \\ \text{a pair of morphisms } \; M \underset{i}{\overset{p}{\rightleftarrows}} \text{Hom}(X, \bullet) \; \text{ with } p \circ i = \text{id}_M. \end{cases}$$

Corollary:

If $\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}$*, then:*

(i) $\mathbb{T}\text{-mod}(\text{Set})_{fp} =$ *Karoubi completion of* $\mathcal{C}^{\text{op}}$.

(ii) $\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{M}^{\text{op}}}$ *for* $\mathcal{M} = \mathbb{T}\text{-mod}(\text{Set})_{fp}$.

(iii) *Any (set-based) model of* $\mathbb{T}$
is a filtering colimit of finitely presentable models.

Application to the computation of classifying toposes

Consider a geometric theory $\mathbb{T}$ on a signature Σ.

$\longrightarrow$ Construct a presheaf-type theory $\mathbb{T}'$ such that $\mathbb{T}$ is a quotient of $\mathbb{T}'$.
Ex: Take for $\mathbb{T}'$ the empty theory on Σ.
(Warning: It is not always a good choice.)

$\longrightarrow$ Consider the (essentially) small category

$$\mathcal{M} = \mathbb{T}'\text{-mod}(\text{Set})_{fp}$$

with $\mathcal{E}_{\mathbb{T}'} \cong \widehat{\mathcal{M}^{\text{op}}}$.

$\longrightarrow$ Compute the (unique) topology J on $\mathcal{M}^{\text{op}}$ such that

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{(\mathcal{M}^{\text{op}})}_J .$$

The problem of characterizing presheaf-type theories:

If we want to compute

the classifying topos $\mathcal{E}_{\mathbb{T}}$

of some geometric theory $\mathbb{T}$,

one tries to write $\mathbb{T}$ as

a quotient of a geometric theory $\mathbb{T}'$

which, hopefully, is presheaf-type.

$\longrightarrow$ Natural question:

How to prove that a geometric $\mathbb{T}'$ is presheaf-type?

The syntactic characterization of presheaf-type theories

For any geometric theory $\mathbb{T}$,
its classifying topos $\mathcal{E}_{\mathbb{T}}$ is associated to

$$\begin{cases} \mathcal{C}_{\mathbb{T}} = \text{ syntactic category of } \mathbb{T}, \\ J_{\mathbb{T}} = \text{ syntactic topology of } \mathcal{C}_{\mathbb{T}} \end{cases}$$

and the canonical functor

$$\ell : \mathcal{C}_{\mathbb{T}} \hookrightarrow \widehat{\mathcal{C}}_{\mathbb{T}} \xrightarrow{j^*} \mathcal{E}_{\mathbb{T}}$$

is fully faithful.

Furthermore:

- objects of $\mathcal{C}_{\mathbb{T}}$ = geometric formulas $\varphi(\vec{x})$ on the signature Σ of $\mathbb{T}$,
- morphisms of $\mathcal{C}_{\mathbb{T}}$ = geometric formulas

$$\theta(\vec{x}, \vec{y}) : \varphi(\vec{x}) \longrightarrow \psi(\vec{y})$$

  which are $\mathbb{T}$-provably functional, up to $\mathbb{T}$-provable equivalence,
- $J_{\mathbb{T}}$-covering families = globally epimorphic families.

Definition:

(i) *An object E of a topos $\mathcal{E}$ is called "irreducible" if any globally epimorphic family*

$$E_i \longrightarrow E\,, \qquad i \in I\,,$$

has a spliting

$$E \longrightarrow E_{i_0} \quad \text{for some } i_0 \in I$$

such that $(E \to E_{i_0} \to E) = \mathrm{id}_E$.

(ii) *If $\begin{cases} \mathcal{C} = \text{(essentially) small category,} \\ J = \text{topology on } \mathcal{C}, \end{cases}$*

an object X of $\mathcal{C}$
is called "irreducible"
if its only J-covering sieve is the maximal sieve.

(iii) *If $\mathbb{T}$ = geometric theory of signature Σ,*
a geometric formula of Σ

$$\varphi(\vec{x})$$

is called "irreducible" if
it is irreducible as an object of $\mathcal{C}_{\mathbb{T}}$
for the topology $J_{\mathbb{T}}$.

Remark:

A geometric formula of Σ

$$\varphi(\vec{x})$$

is $J_{\mathbb{T}}$-irreducible in $\mathcal{C}_{\mathbb{T}}$ if and only if:

For any family of $\mathbb{T}$-provably functional formulas

$$\varphi_i(\vec{x}_i) \xrightarrow{\theta_i(\vec{x}_i,\vec{x})} \varphi(\vec{x}) \,, \quad i \in I \,,$$

such that $\varphi(\vec{x}) \vdash_{\vec{x}} \bigvee_{i \in I} (\exists \, \vec{x}_i) \, \theta_i(\vec{x}_i, \vec{x})$ is $\mathbb{T}$-provable,

there exist

$$\begin{cases} \bullet \quad \text{an index } i_0 \in I, \\ \bullet \quad \text{a } \mathbb{T}\text{-provably functional formula} \\ \qquad \varphi(\vec{x}) \xrightarrow{\theta(\vec{x},\vec{x}_{i_0})} \varphi_{i_0}(\vec{x}_{i_0}) \end{cases}$$

such that the formula

$$(\exists \, \vec{x}_{i_0})(\theta(\vec{x}, \vec{x}_{i_0}) \wedge \theta_{i_0}(\vec{x}_{i_0}, \vec{x}\,'))$$

is $\mathbb{T}$-provably equivalent to the formula

$$\vec{x} = \vec{x}\,' \,.$$

Lemma: Considering the bridge

irreducible objects of
$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{C}}$

$(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ $\qquad\qquad$ $(\mathcal{C}, \mathrm{Disc})$

we have:

(i) *If $\mathcal{E} = \widehat{\mathcal{C}}$, irreducible objects of $\mathcal{E}$ are splitings of representable objects $\mathrm{Hom}(\bullet, X)$, $X =$ object of $\mathcal{C}$.*

(ii) *If $\mathcal{E} = \mathcal{E}_{\mathbb{T}}$, irreducible objects of $\mathcal{E}$ are irreducible formulas $\varphi(\vec{x})$.*

Remarks:

(i) If $\mathcal{E} = \widehat{\mathcal{C}}$, the Karoubi completion of $\mathcal{C}$ is the full subcategory of $\mathcal{E}$ on irreducible objects.

(ii) If $\mathbb{T}$ is presheaf-type, there is a one-to-one correspondence

$$\left\{\begin{matrix}\text{irreducible}\\ \text{formulas}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{finitely presentable}\\ \text{set-based models of } \mathbb{T}\end{matrix}\right\}.$$

Theorem (Caramello):

Let $\mathbb{T}$ = geometric theory on a signature Σ.

Then $\mathbb{T}$ is presheaf-type

if and only if

any geometric formula of Σ

$$\psi(\vec{y}),$$

considered as an object of $\mathcal{C}_{\mathbb{T}}$,

has a $J_{\mathbb{T}}$-covering

$$\theta_i(\vec{x}_i, \vec{y}) : \varphi_i(\vec{x}_i) \longrightarrow \psi(\vec{y})$$

by irreducible formulas

$$\varphi_i(\vec{x}_i).$$

Another criterion for being presheaf-type: the equivalence of syntax and semantics

Theorem (Caramello):

Let $\mathbb{T}$ = geometric theory on a signature Σ.

Then $\mathbb{T}$ is presheaf-type if and only if
the following 3 conditions are verified:

(1) *For any geometric sequent of Σ*

$$\varphi \vdash_{\vec{x}} \psi \,,$$

it is $\mathbb{T}$-provable if and only if
it is verified by all set-based models.

(2) *Any finitely presentable set-based model*

$$M = \text{object of } \mathbb{T}\text{-mod}\,(\text{Set})_{fp}$$

is finitely presented

by some (irreducible) geometric formula

$$\varphi\,(x_1^{A_1} \cdots x_n^{A_n})$$

in the sense that,

for any set-based model N,

there is a natural one-to-one correspondence

$$\left\{\begin{matrix} \textit{model morphisms} \\ M \to N \end{matrix}\right\} \longleftrightarrow \begin{cases} \textit{families of elements} \\ (x_1^{A_1}, \cdots, x_n^{A_n}) \in NA_1 \times \cdots \times NA_n \\ \textit{which verify the formula} \\ \qquad\qquad \varphi\,(x_1^{A_1}, \cdots, x_n^{A_n})\,. \end{cases}$$

(3) *For any sorts* $A_1, \cdots, A_n$ *of* Σ *and any map*

$$P : \begin{array}{c} M \\ \| \\ \text{finitely presentable} \\ \text{set-based model of } \mathbb{T} \end{array} \longmapsto \begin{array}{c} MP \\ \| \\ \text{subset of} \\ MA_1 \times \cdots \times MA_n \end{array}$$

such that, any model morphism

$$u : M \longrightarrow N$$

induces a commutative square

$$\begin{array}{ccc} MP & \longrightarrow & NP \\ \downarrow & & \downarrow \\ MA_1 \times \cdots \times MA_n & \xrightarrow{u_{A_1} \times \cdots \times u_{A_n}} & NA_1 \times \cdots \times NA_n \end{array}$$

then P *is defined by a geometric formula*

$$\varphi(x_1^{A_1} \cdots x_n^{A_n}) .$$

Proof in the direct sense:

(1) follows from the equivalence

$$\widehat{(\mathcal{C}_{\mathbb{T}})}_{J_{\mathbb{T}}} \cong \mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{M}^{\mathrm{op}}}$$

with $\mathcal{M} = \mathbb{T}\text{-mod}\,(\mathrm{Set})_{fp}$.

(2) follows from the bridge:

irreducible objects of
$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{M}^{\mathrm{op}}}$

$(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ $\qquad$ $(\mathcal{M}^{\mathrm{op}}, \mathrm{Disc})$

(3) follows from the bridge

subobjects of
$U_{\mathbb{T}} A_1 \times \cdots \times U_{\mathbb{T}} A_n$

$(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ $\qquad$ $(\mathcal{M}^{\mathrm{op}}, \mathrm{Disc})$

(for $U_{\mathbb{T}}$ = universal model of $\mathbb{T}$ in $\mathcal{E}_{\mathbb{T}}$).

Application to the computation of classifying toposes:

Suppose that $\mathbb{T}'$ is a quotient of $\mathbb{T}$,
$\mathbb{T}$ is a presheaf-type theory and so

$$\mathcal{E}_{\mathbb{T}} \cong \widehat{\mathcal{M}^{\mathrm{op}}}$$

for $\mathcal{M} = \mathbb{T}\text{-mod}\,(\mathrm{Set})_{fp}$,
$\mathcal{M}^{\mathrm{op}} \cong$ full subcategory of $\mathcal{C}_{\mathbb{T}}$ on irreducible formulas.

Corollary: We have

$$\mathcal{E}_{\mathbb{T}'} \cong \widehat{(\mathcal{M}^{\mathrm{op}})}_J$$

if J is the topology on $\mathcal{M}^{\mathrm{op}}$ defined in the following way:

A family of morphisms between irreducible formulas

$$\theta_i(\vec{x}_i, \vec{x}) : \varphi_i(\vec{x}) \longrightarrow \varphi(\vec{x}), \quad i \in I,$$

(corresponding to a family of morphisms between finitely presented models

$$M \longrightarrow M_i, \quad i \in I)$$

is J-covering if and only if the sequent

$$\varphi(\vec{x}) \vdash_{\vec{x}} \bigvee_{i \in I} (\exists\, \vec{x}_i)\, \theta_i(\vec{x}_i, \vec{x})$$

is $\mathbb{T}'$-provable.