A derived Gabriel-Popescu theorem for t-structures via derived injectives (joint w/ Julia Ramos González)

I) LINEAR TOPOI

- categories $\rightsquigarrow$ ($\mathbb{Z}$-) linear categories
- presheaves $\rightsquigarrow$ modules

Let $\underline{a}$ be a linear category.

$\rightsquigarrow$ Abelian category $\mathrm{Mod}(\underline{a})$ = linear functors $\underline{a}^{op} \to \underbrace{\mathrm{Mod}(\mathbb{Z})}_{\text{abelian groups}}$

- Question: what is a linear topos?

  Answer: a GROTHENDIECK ABELIAN CATEGORY

- Key result: the GABRIEL-POPESCU THEOREM

  → Let $\mathcal{A}$ be a Grothendieck abelian category, and let $\jmath : \mathcal{U} \hookrightarrow \mathcal{A}$ be the full subcategory spanned by the generators. Then, the functor

$$G : \mathcal{A} \longrightarrow \mathrm{Mod}(\mathcal{U})$$
$$A \longmapsto \mathcal{A}(\jmath(-), A)$$

Mitchell 1981

⟸ G preserves injectives

is fully faithful and has an EXACT left adjoint.

SLOGAN: Grothendieck ab. categories are exact localizations of module categories

II) GO DERIVED! (TOWARDS "LINEAR DERIVED/HIGHER TOPOI")

- Want: "higher counterpart" of (Grothendieck) abelian categories
  - ↳ In particular, a higher counterpart of $\mathrm{Mod}(\underline{a})$

<u>Idea</u>: replace "abelian group" with "complex of abelian groups"
work up to <u>quasi-isomorphism</u>

↳ $\underline{a}$ linear category $\rightsquigarrow$ $\underline{a}$ dg-category ($\to H^0(\underline{a})$ lin. cat
0th. cohomology)

↳ $\mathrm{Mod}(\underline{a})$ $\rightsquigarrow$ $D(\underline{a})$ "dg-functors" $\underline{a}^{op} \to \mathrm{dg\,Mod}(\mathbb{Z})$

"derived (dg) category"

<u>Observe</u>:
- if $V \in D(\underline{a})$, we have the n-SHIFT $V[n]$
- if $V \xrightarrow{f} W$ in $D(\underline{a})$, we can form a DISTINGUISHED TRIANGLE:

$$V \xrightarrow{f} W \to C(f) \to V[1]$$

- A dg-category $\mathcal{A}$ is PRETRIANGULATED if it is closed under shifts and distinguished triangles.

- Porta 2010: a GABRIEL-POPESCU THEOREM FOR PRETRIANGULATED DG-CATEGORIES (with suitable completeness and generators).

- Still, pretriangulated dg-categories, by themselves, do not look really like the "higher counterpart" of abelian categories.

  ↳ What is an injective object?

- Assume that $\underline{a}$ is CONCENTRATED IN NONPOSITIVE DEGREES.

  - $V \in D(\underline{a}) \rightsquigarrow$ TRUNCATIONS: $\tau_{\leq n} V \in D(\underline{a})_{\leq n}$
    $\tau_{\geq n} V \in D(\underline{a})_{\geq n}$

  $\rightarrow$ COHOMOLOGY:

  - $H^0(V) = \tau_{\leq 0}\tau_{\geq 0} V \in \mathrm{Mod}(H^0(\underline{a}))$
  - $H^n(V) = H^0(V[n])$

  $D(\underline{a})_{\leq 0} \cap D(\underline{a})_{\geq 0} \cong \mathrm{Mod}(H^0(\underline{a}))$

- A T-STRUCTURE on a pretriangulated dg-category $\mathcal{A}$ (actually, on $H^0(\mathcal{A})$) is additional data which yields:

  - truncations of objects $\tau_{\leq n} A \in \mathcal{A}_{\leq n}$, $\tau_{\geq n} A \in \mathcal{A}_{\geq n}$
  - cohomology of objects: $H^0(A) = \tau_{\leq 0}\tau_{\geq 0} A$,

with values in the HEART $\mathcal{A}^\heartsuit = \mathcal{A}_{\leq 0} \cap \mathcal{A}_{\geq 0}$, ← ABELIAN CATEGORY!

(Abuse of notation: $\mathcal{A} \rightsquigarrow H^0(\mathcal{A})$)

- This is the right setting!

- DERIVED INJECTIVE OBJECTS:
  - $E \in \mathcal{A}_{\geq 0}$
  - $H^0(E)$ is injective in $\mathcal{A}^\heartsuit$.
  - $H^0(\mathcal{A})(A, E) \xrightarrow[H^0]{\sim} \mathcal{A}^\heartsuit(H^0(A), H^0(E))$ (for all $A \in \mathcal{A}$)

- They behave nicely:
  - can do "derived injective resolutions" (cf. work w/ Wendy Lowen, Michel Van den Bergh)
  - maybe, a proof of a Gabriel-Popescu theorem? YES!

SLOGAN: pretr. dg-cats w/ "Grothendieck t-structures" ($\rightarrow$ heart is Grothendieck + "generators") are "LINEAR HIGHER TOPOI"

III) A GABRIEL-POPESCU THEOREM FOR T-STRUCTURES

<u>Theorem</u>:

Let $\mathcal{A}$ be a pretriangulated dg-category with a "Grothendieck t-structure".

Let $\mathcal{U}$ be a set of generators in $\mathcal{A}_{\leq 0}$

(every $A \in \mathcal{A}_{\leq 0}$ admits a map $\bigoplus U_i \xrightarrow{p} A$ with $U_i \in \mathcal{U}$, $H^0(p)$ an epimorphism in $\mathcal{A}^\heartsuit$)

Take $\underline{u} = \tau_{\leq 0}\mathcal{U}$, $J : \underline{u} \to \mathcal{A}$.

$$\begin{aligned} G : \mathcal{A} &\longrightarrow D(\underline{u}) \\ A &\longmapsto \mathcal{A}(J(-), A). \end{aligned}$$

G preserves derived injectives

Then, G is fully faithful and has a <u>t-exact</u> left adjoint

<u>SLOGAN</u>: every pretr. dg-cat w/ Grothendieck t-structure is a t-exact localization of a derived category of a dg-category (in ≤ 0)

Final remarks:

- Lurie in "Spectral Algebraic Geometry" (Appendix C) has similar results using PRESTABLE ∞-CATEGORIES.
  (our proof is new and based on Mitchell (1981) and derived injectives)

- Next?
  ↳ Understand TENSOR PRODUCTS of t-structures
  ↳ Understand "linear derived sites/topologies").

THANKS!