Towards Higher Topology

Ivan Di Liberti

Toposes online
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Czech Academy of Sciences

This talk is based on the paper:

1. **Towards Higher Topology**, ArXiv:2009.14145, soon on JPAA.

For the full story, check out:

2. **General facts on the Scott Adjunction**, ArXiv:2009.14023.
3. **Formal Model Theory & Higher Topology**, ArXiv:2010.00319.
4. **The Scott Adjunction**, ArXiv:2009.07320.

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Sketches of an elephant

These cover three different aspects of the same story.

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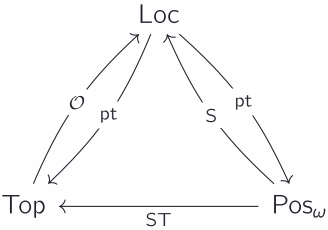
Today we will focus only on the geometric side of the story.

The topological picture



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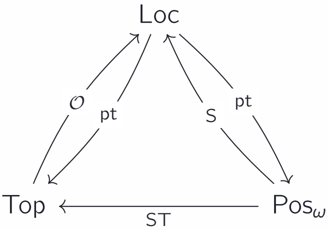

Top is the category of topological spaces and continuous mappings between them.

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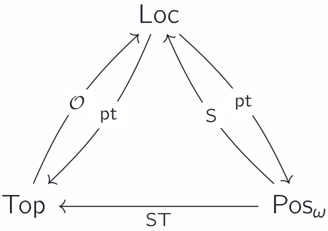

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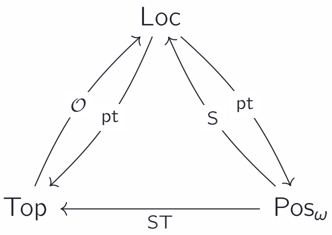

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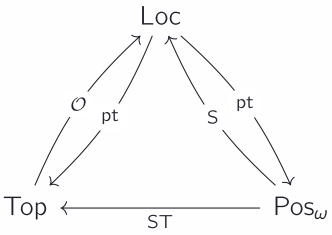

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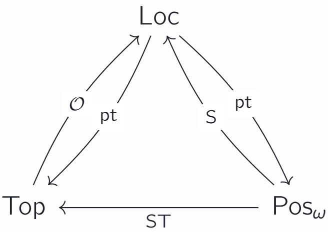

Loc is the category of Locales. It is defined to be the opposite category of frames, where objects are frames and morphisms are morphisms of frames.

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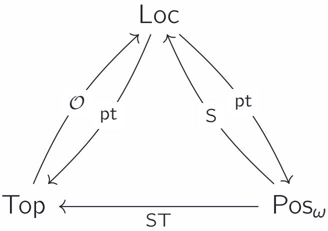

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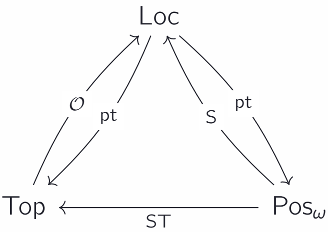

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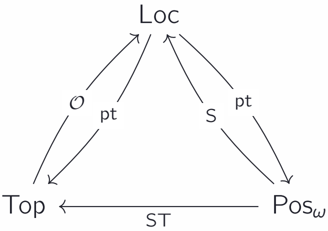

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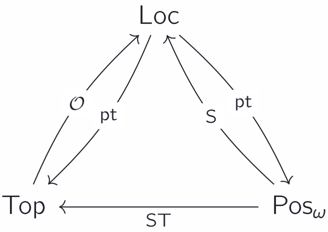

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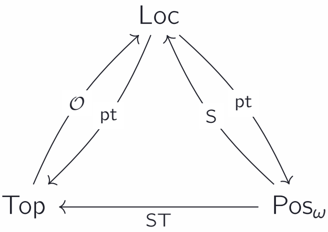

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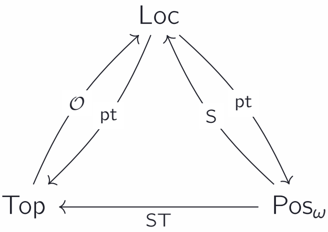

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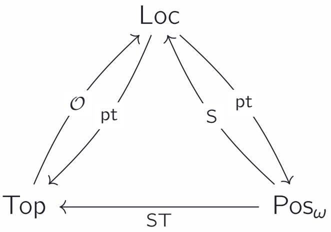

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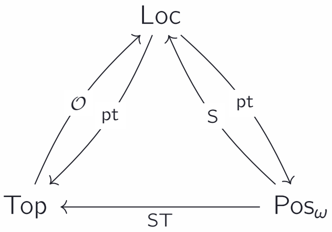

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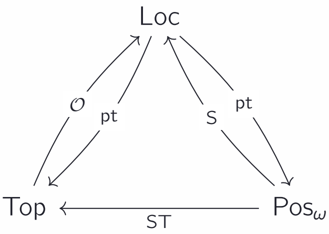

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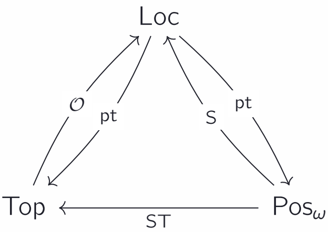

pt maps in both cases a locale to its set of *formal* points. A formal point of a locale $\mathcal{L}$ is a morphism of locales $\mathbb{T} \to \mathcal{L}$. This set admits a topology, but also a partial order.

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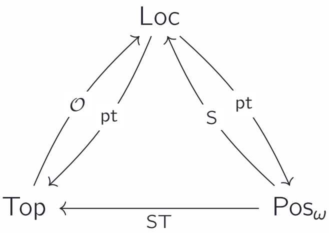

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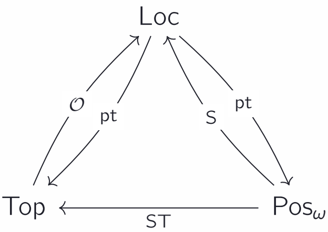

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ST equips a poset P with its Scott topology, which can be essentially identified with the frame above.

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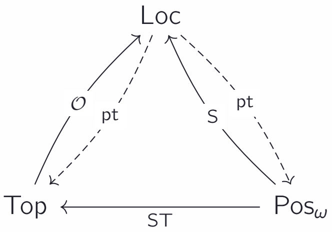

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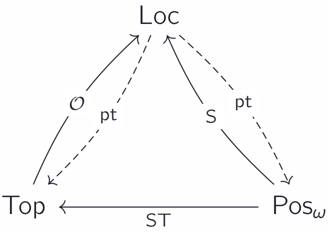

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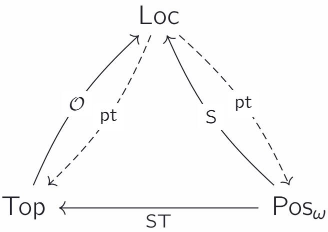

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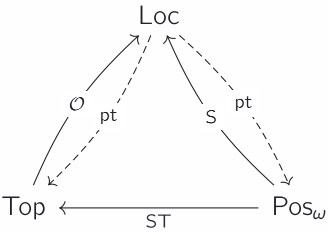

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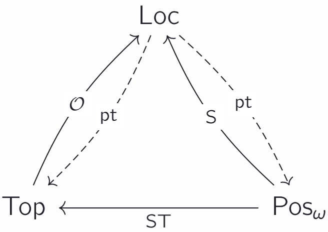

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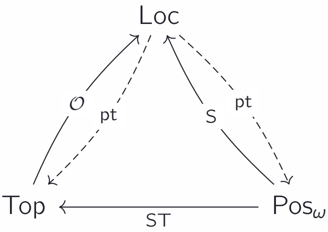

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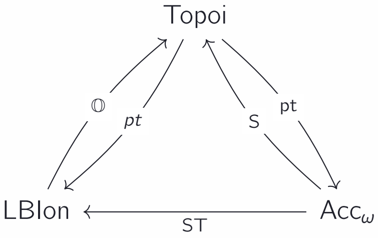

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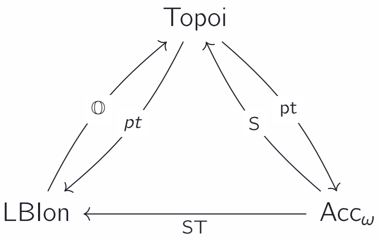

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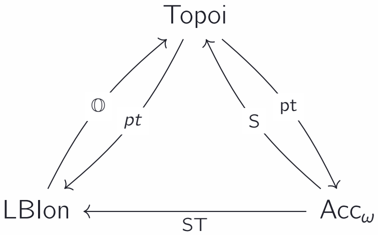

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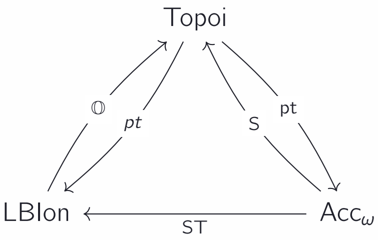

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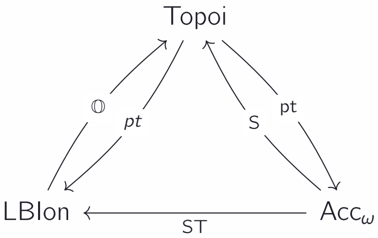

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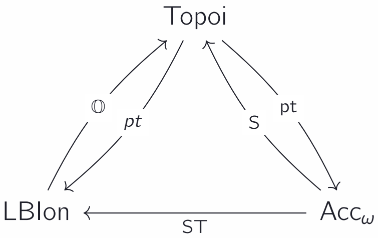

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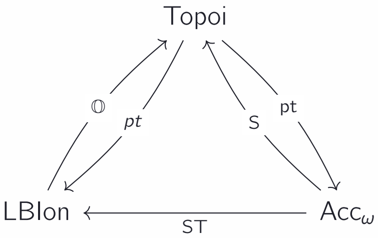

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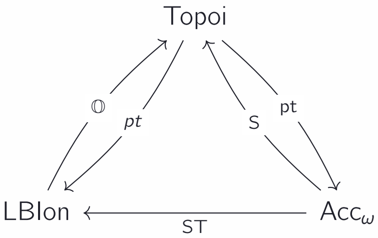

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Ionads

The 2-category of Ionads was introduced by Garner. A **ionad** $\mathcal{X} = (X, \mathsf{Int})$ is a set X together with a comonad $\mathsf{Int} : \mathsf{Set}^X \to \mathsf{Set}^X$ preserving finite limits. While topoi are the categorification of locales, Ionads are the categorification of the notion of topological space, to be more precise, Int categorifies the interior operator of a topological space.

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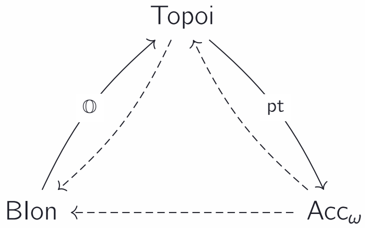
Thm. (Garner)

The category of coalgebras for a ionad is indicated with $\mathbb{O}(\mathcal{X})$ and is a cocomplete elementary topos. A ionad is bounded if $\mathbb{O}(\mathcal{X})$ is a Grothendieck topos. Thus one should look at the functor

$$\mathbb{O} : \mathsf{BIon} \to \mathsf{Topoi},$$

as the categorification of the functor that associates to a space its frame of open sets.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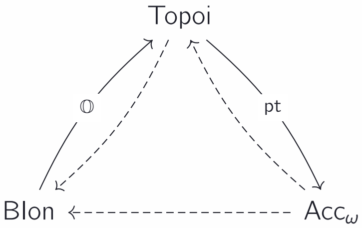
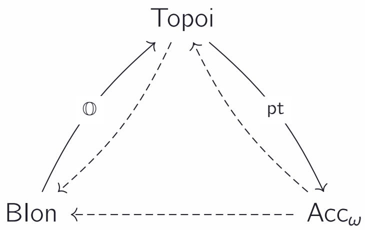


1. The functor pt was also known to the literature. For every topos $\mathcal{E}$ one can define its category of points to be $\mathsf{Topoi}(\mathsf{Set}, \mathcal{E})$, and it is a classical result that this category is accessible and has directed colimits.



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2. My task was to provide all the dashed arrows in this diagram, to show that they form adjunctions and to describe their properties.

The Scott Adjunction (Henry, DL)

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2. Topoi is the 2-category of Groethendieck topoi. A 1-cell is a geometric morphism and has the direction of the right adjoint. 2-cells are natural transformation between left adjoints.

The Scott construction

Let $\mathcal{A}$ be a 0-cell in Acc_ω. $\mathsf{S}(\mathcal{A})$ is defined as the category $\mathsf{Acc}_\omega(\mathcal{A}, \mathsf{Set})$.

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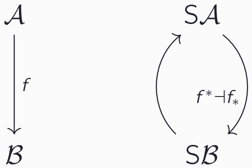
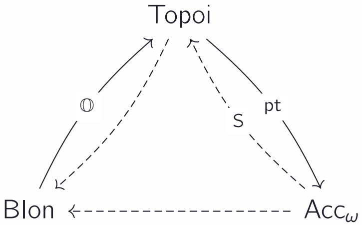
Let $\mathcal{A}$ be a 0-cell in Acc_ω. $\mathsf{S}(\mathcal{A})$ is defined as the category $\mathsf{Acc}_\omega(\mathcal{A}, \mathsf{Set})$.Let $f : \mathcal{A} \to \mathcal{B}$ be a 1-cell in Acc_ω.



$\mathsf{S}f = (f^* \dashv f_*)$ is defined as follows: f^* is the precomposition functor $f^*(g) = g \circ f$. This is well defined because f preserve directed colimits. f^* preserve all colimits and thus has a right adjoint, that we indicate with f_*. Observe that f^* preserve finite limits because finite limits commute with directed colimits in Set.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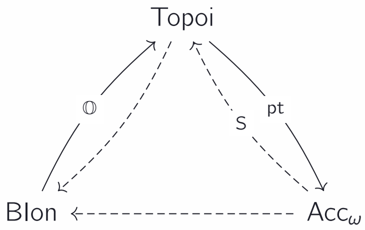

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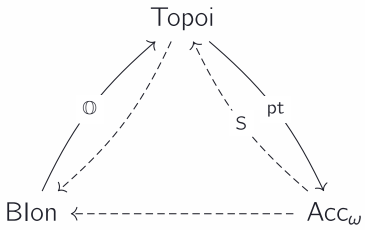

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   In order to fix this problem, one needs to stretch Garner's definition and introduce **generalized (bounded) Ionads**.

Generalized Ionads

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The result above allows to produce comonads on $\P(X)$ (just compose $f^* f_*$) and follows from the general theory of total categories, but needs $\P(X)$ to be locally small to stay in place. Thus the choice of Set^X would have generated size issues. A similar issue would arise with Kan extensions.

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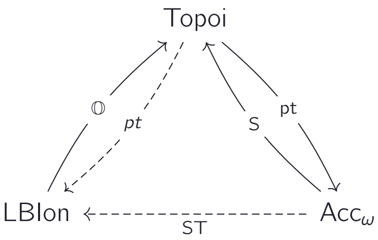
Exa.

Accessible ultracategories à la Makkai (or more recently Lurie) admit a natural structure of (compact) ionad.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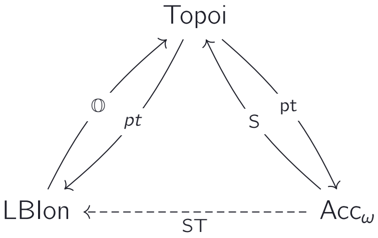
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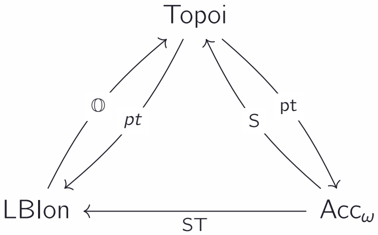
Thm. (DL)

Replacing bounded Ionads with generalized bounded Ionads, there exists a right adjoint for $\mathbb{O}$ and a Scott topology-construction ST such that $\mathsf{S} = \mathbb{O} \circ \mathsf{ST}$, in complete analogy to the posetal case.



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Toposes online
Jun 2021