Higher Sheaves

Georg Biedermann
Universidad del Norte, Barranquilla, Colombia

Joint work with Mathieu Anel, Eric Finster and André Joyal

Toposes online, 28.6.2021

Articles

Results presented here are contained in

Higher Sheaves and Left-Exact Localizations in ∞-Topoi
Mathieu Anel, GB, Eric Finster, André Joyal,
https://arxiv.org/abs/2101.02791

Some results also contained in

Modalities in Homotopy Type Theory
Rijke, Shulman, Spitters
Logical Methods in Computer Science, 16(1), 2020, pp. 2:1–2:79

1-topoi

Basic facts about 1-topoi:

- The category of presheaves $\mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ on a small category $\mathscr{C}$ with values in sets is a 1-topos.
- *Every* 1-*topos* is the left exact localization of such a presheaf topos.
- Any left exact localization of a presheaf topos $\mathrm{Set}^{\mathscr{C}^{\mathrm{op}}}$ is the sheafification with respect to a Grothendieck topology on $\mathscr{C}$.
- Every left exact localization of a 1-topos is accessible.

∞-topoi

The category $\mathscr{C}$ is an $(\infty, 1)$-category.

The category $\mathscr{S}$ is the category of ∞-groupoids. An object in $\mathscr{S}$ is a *space* (homotopy type, type, anima, ...).

Basic facts about ∞-topoi:

- The category of presheaves $\mathscr{S}^{\mathscr{C}^{\mathrm{op}}}$ on a small category $\mathscr{C}$ with values in the category $\mathscr{S}$ of spaces is an ∞-topos.

∞-topoi

Basic facts about ∞-topoi:

- The category of presheaves $\mathscr{S}^{\mathscr{C}^{\mathrm{op}}}$ on a small category $\mathscr{C}$ with values in the category $\mathscr{S}$ of spaces is an ∞-topos.
- *Every ∞-topos* is the accessible left exact localization of such a presheaf topos.
- **Not every** left exact localization of a presheaf topos $\mathscr{S}^{\mathscr{C}^{\mathrm{op}}}$ is sheafification with respect to a Grothendieck topology on $\mathscr{C}$.

Questions we need to answer

Given a small category $\mathscr{C}$, how to describe all left exact localizations of $\mathscr{S}^{\mathscr{C}^{\mathrm{op}}}$?

Given an ∞-topos $\mathscr{E}$, how to describe all left exact localizations of $\mathscr{E}$?

Given a set S of maps in an ∞-topos $\mathscr{E}$, how can we invert S left exactly?

What is a sheaf with respect to S?

Several reflective subcategories

Given a set S of morphisms in an ∞-topos, we will define a sequence of reflective subcategories of $\mathscr{E}$:

$$\mathrm{Sh}(\mathscr{E}, S) \subset \mathrm{Mod}(\mathscr{E}, S) \subset \mathrm{Loc}(\mathscr{E}, S) \subset \mathscr{E}.$$

Local objects

Definition

Let $\mathscr{E}$ be an ∞-topos and S a set of maps in $\mathscr{E}$. An object X in $\mathscr{E}$ is *S-local* if the map

$$f^* : \mathrm{map}_{\mathscr{E}}(B, X) \xrightarrow{\simeq} \mathrm{map}_{\mathscr{E}}(A, X)$$

is a weak equivalence for every $f : A \to B$ in S. We write $\mathrm{Loc}(\mathscr{E}, S)$ for the full subcategory of $\mathscr{E}$ given by the S-local objects.

Note that $\mathrm{Loc}(\mathscr{E}, S)$ is presentable and reflective in $\mathscr{E}$.

Modal objects

Definition (ABFJ)

Let S be a set of maps in an ∞-topos $\mathscr{E}$. An object X in $\mathscr{E}$ is *S-modal* if it is local with respect to every base change of the maps in S. We write $\mathrm{Mod}(\mathscr{E}, S)$ for the full subcategory of $\mathscr{E}$ given by the S-modal objects.

Remark

Note:

- $\mathrm{Mod}(\mathscr{E}, S)$ is presentable and reflective in $\mathscr{E}$.
- If $\mathcal{G}$ is a set of generators for $\mathscr{E}$, then in the above definition it suffices to consider only base changes along maps having their target in $\mathcal{G}$.

Diagonals

Definition

The *diagonal* of a map $f : A \to B$ in $\mathscr{E}$ is the map

$$\Delta f : A \to A \times_B A.$$

For $n \geq 0$ we define:

$$\Delta^0 f = f \quad \text{and} \quad \Delta^n f = \Delta(\Delta^{n-1} f).$$

Example:

- $\Delta(A \to *) = A \to A \times A$ and $\Delta^n(A \to *) = A \to A^{S^{n-1}}$

Definition

Let S be a set of maps in $\mathscr{E}$. The *diagonal closure* of S is the set

$$\Delta^\infty S = \{\Delta^n f \mid f \in S, n \geq 0\}.$$

Higher Sheaves

Definition (ABFJ)

Let S be a set of maps in an ∞-topos $\mathscr{E}$. An object X of $\mathscr{E}$ is an S-sheaf if it is modal with respect the set $\Delta^{\infty}(S)$. The full subcategory of $\mathscr{E}$ given by the S-sheaves is denoted by $\mathrm{Sh}(\mathscr{E}, S)$.

Theorem (ABFJ)

The category $\mathrm{Sh}(\mathscr{E}, S)$ is presentable and reflective in $\mathscr{E}$. The reflector

$$L : \mathscr{E} \to \mathrm{Sh}(\mathscr{E}, S)$$

is a left exact localization. In particular, $\mathrm{Sh}(\mathscr{E}, S)$ is an ∞-topos. The functor L is initial among all left exact and cocontinuous functors that invert the maps in S.

Higher Sites

In the category $\mathscr{S}^{\mathscr{C}^{\mathrm{op}}}$ we write $R(\mathscr{C})$ for the set of representable functors

$$R_C = \mathrm{map}_{\mathscr{C}}(-, C) : \mathscr{C}^{\mathrm{op}} \to \mathscr{S} \quad \text{for all} \quad C \in \mathscr{C}.$$

Definition (ABFJ)

An ∞-*site* is a pair $(\mathscr{C}, S)$ where $\mathscr{C}$ is an $(\infty, 1)$-category and S is a set of maps in $\mathscr{S}^{\mathscr{C}^{\mathrm{op}}}$.
A *sheaf with respect to the* ∞-*site* $(\mathscr{C}, S)$ is a presheaf $X : \mathscr{C}^{\mathrm{op}} \to \mathscr{S}$ which is local with respect to all $R(\mathscr{C})$-base changes of the maps in $\Delta^{\infty}(S)$.

Corollary

Every ∞*-topos is equivalent to an* ∞*-topos of sheaves on an* ∞*-site.*

Monomorphisms

Definition

A map $f : A \to B$ is called a *monomorphism* if its diagonal

$$\Delta f : A \to A \times_B A$$

is an isomorphism.

Note:

- monomorphism $=$ (-1)-truncated map
- In $\mathscr{S}$: monomorphism $=$ inclusion of connected components
- In sets: monomorphism $=$ injection.

Topological localizations

Definition (Lurie)

Let $\mathscr{E}$ be an ∞-topos and let $L : \mathscr{E} \to \mathrm{Sh}(\mathscr{E}, S)$ be the left exact localization that universally inverts the set S of maps of $\mathscr{E}$. If S consists only of monomorphisms in $\mathscr{E}$ then L is called *topological*.

Theorem (Lurie)

Topological localizations on $\mathscr{S}^{\mathscr{C}^{\mathrm{op}}}$ correspond bijectively to Grothendieck topologies on $\mathscr{C}$.

The classical sheaf condition

Corollary

If the set S in the ∞-topos $\mathscr{S}^{\mathscr{C}^{\mathrm{op}}}$ consists only of monomorphisms then

$$\mathrm{Mod}(\mathscr{E}, S) = \mathrm{Sh}(\mathscr{E}, S)$$

and our sheaf condition reduces to the classical sheaf condition given by Grothendieck topologies.

Proof.

f monomorphism $\iff$ Δf isomorphism
Hence $\Delta^{\infty} f = \{f\} \cup \{isos\}$ □

In 1-topoi any lex localization is topological

Let $f : A \to B$ be a map in a 1-topos. We would like to invert f left exactly.

Factor f into a surjection g followed by a monomorphism h:

$$A \xrightarrow{g} \mathrm{im}(f) \xrightarrow{h} B.$$

If we want to invert f we can

- first invert the monomorphism h and all its base changes, (makes f surjective)
- and then invert $L_h(\Delta f) = \Delta(L_h f) = \Delta(L_h g)$ and all its base changes. (makes f mono)

In any 1-category the diagonal of *any* map is a monomorphism!

In ∞-topoi not

Let A be a space and $f : A \to *$. What are the fibers of $\Delta f = \Delta A$?

$$\begin{array}{ccc} A & \xrightarrow{\simeq} & P(A) = \{\gamma : [0,1] \to A\} \\ {\scriptstyle \Delta A}\downarrow & \swarrow{\scriptstyle (\gamma(0),\gamma(1))} & \\ A \times A & & \end{array}$$

Fiber:

$$\Delta(A)_{(a,b)} = \{\gamma \mid \gamma(0) = a, \gamma(1) = b\}$$

$$\Delta(A)_{(a,a)} = \Omega_a A$$

Hence, ΔA is *not* a monomorphism.

Cotopological localizations

Definition (Lurie)

Let $\mathscr{E}$ be a topos and L be a left exact localization on $\mathscr{E}$. Then L is called *cotopological* if L inverts *no* monomorphism.

Theorem (Lurie)

Any left exact localization L of a topos $\mathscr{E}$ can be factored into a topological localization L_{top} followed by a cotopological one L_{cotop}:

$$\mathscr{E} \xrightarrow{\;L\;} \mathscr{G}, \qquad \mathscr{E} \xrightarrow{\;L_{\mathrm{top}}\;} \mathscr{F} \xrightarrow{\;L_{\mathrm{cotop}}\;} \mathscr{G}$$

Cotopological localizations, so what?

Example

Let Fin denote the category of finite spaces. The category of functors $\mathscr{S}^{\mathrm{Fin}} = \mathscr{S}[X]$ is a topos. Evaluation at the terminal space $*$ is a left exact localization

$$\mathscr{S}[X] \xrightarrow{P_0} \mathscr{S}\ ,\ F \mapsto F(*).$$

This localization P_0 has a nontrivial cotopological part!

Forthcoming paper: Goodwillie calculus “happens” in the cotopological part.

Cotopological localizations are at the center of (unstable) homotopy theory!

Example elaborated

The functor X in $\mathscr{S}[X]$ is the inclusion $X : \mathrm{Fin} \hookrightarrow \mathscr{S}$. Consider the map

$$R^* = X \to * = R^{\emptyset}.$$

What is

- $\mathrm{Loc}(\mathscr{S}[X], X \to *)$?
  F is X-local if and only if $F(\emptyset) \xrightarrow{\simeq} F(*)$.
- $\mathrm{Mod}(\mathscr{S}[X].X)$?
  Base changes: $R^{K \sqcup *} = X \times R^K \to R^K$
  F is X-modal if and only if $F(K) \xrightarrow{\simeq} F(K \sqcup *)$.
- $\mathrm{Sh}(\mathscr{S}[X], X)$?
  $\Delta^{\infty}(X \to *) = \{X \to X^{S^{n-1}} = R^{S^{n-1}} \mid n \geq 0\}$
  Base changes: Yoneda image of cofiber sequence $S^{n-1} \to K \to K \cup D^n$
  F is an X-sheaf if and only if $F(K) \xrightarrow{\simeq} F(K \cup D^n)$.
  Thus, F is constant.

A new construction for the Goodwillie tower

Theorem (ABFJ)

Goodwillie tower is given in the following way: