Ranges of functors and elementary classes via topos theory

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$\mathbb{S}, \mathbb{T}$ first order theories, $F\colon \mathcal{A} := Mod(\mathbb{T}) \to Mod(\mathbb{S}) =: \mathcal{B}$ a functor.

Questions:

1. (i) Is every $B \in \mathcal{B}$ isomorphic to $F(A)$ for some $A \in \mathcal{A}$?
2. (ii) Is every $B \in \mathcal{B}$ elementarily equivalent to $F(A)$ for some $A \in \mathcal{A}$?
3. (iii) What is the $\underbrace{\text{elementary class generated by the essential image of } F}_{:= Mod(\bigcap_{A \in \mathcal{A}} Th(F(A)))}$?

Rephrasing of (iii): Is there some first order statement true for every $F(A)$, but not for all Bs ?

Examples

Example 1: Dilworth's congruence lattice problem

$$\begin{array}{rcl}\mathbf{Lat} & \to & \mathbf{AlgDistLat} \\ L & \mapsto & Con(L)\end{array}$$

Question (Dilworth 1940s) **:** Is every AlgDistLat of the form $Con(L)$?

Examples

Example 1: Dilworth's congruence lattice problem

$$\begin{array}{rcl} \mathbf{Lat} & \to & \mathbf{DistSemLat} \\ L & \mapsto & Con_c(L) \end{array}$$

Question: Is every DistSemLat of the form $Con_c(L)$?

Examples

Example 1: Dilworth's congruence lattice problem

$$\begin{array}{ccc} \mathbf{Lat} & \to & \mathbf{DistSemLat} \\ L & \mapsto & Con_c(L) \end{array}$$

Question: Is every DistSemLat of the form $Con_c(L)$?

Theorem (Huhn 1985)**:** Every DistSemLat of cardinality $\leq \aleph_1$ is of the form $Con_c(L)$.

Theorem (Wehrung/Růžička 2007/8)**:** There are DistSemLats of cardinality $\geq \aleph_2$ *not* of the form $Con_c(L)$.

Corollary: $\nexists$ first order sentence holding for all $Con_c(L)$, but not for general DistSemLats.

Examples

My motivating examples:

– Representation problem for special groups (quadratic form theory)

– Which graded $\mathbb{Z}/2$-algebras arise as Milnor K-theory of a field? (possible applications to inverse Galois problem)

What are *your* examples?

Some fragments of $L_{\infty\kappa}$

Definition: Let κ be a regular cardinal, Σ a first order signature.

1. A *κ-geometric formula* is a formula built from atomic formulas, $\top, \bot$, using $\bigvee_{j \in J}$ (J a set), $\bigwedge_{i \in I}$ ($|I| < \kappa$) and $\exists \{x_i\}_{i \in I}$ ($|I| < \kappa$).
2. A *κ-geometric theory* is a theory which can be axiomatized by formulas of the form $\forall \{x_i\}\ \phi \to \psi$, where ϕ, ψ are κ-geometric formulas. (*κ-geometric sequents*)
3. For a class of Σ-structures $\mathcal{C}$ denote by $Th_{\kappa\text{-geom}}(\mathcal{C})$ the *κ-geometric theory of $\mathcal{C}$*, i.e. the set of all κ-geometric sequents that are valid in every member of $\mathcal{C}$.
4. Denote by $Th_{\neg\text{-}\kappa\text{-geom}}(\mathcal{C})$ the set of negations of κ-geometric formulas (i.e. sequents of the form $\forall \bar{x}\ \phi \to \bot$), that are valid in every member of $\mathcal{C}$.

Accessible categories

Reminder: Let κ be a regular cardinal.

(i) An object A is *κ-presentable* if for all κ-directed diagrams D

$$Hom(A, colim\ D) \cong colim_{d \in Ob\,D}\ Hom(A, d)$$

(ii) A category is *κ-accessible* if all κ-directed colimits exist and there is a *set* of κ-presentable objects, s.t. every object is a κ-directed colimit of these.

Examples:

– Any variety is $\aleph_0$-accessible. (“finitely accessible”)
– **Fields** is $\aleph_0$-accessible
– **Theorem:** The category $Mod(\mathbb{T})$ for a geometric theory $\mathbb{T}$ is accessible.
– **Theorem:** The κ-accessible categories are exactly the ones of the form $Mod(\mathbb{T})$ for a κ-geometric theory $\mathbb{T}$.

The result

Theorem (A.): Let $\mathcal{A}$, $\mathcal{B}$ be κ-accessible categories, $\mathcal{A} = Mod(\mathbb{T})$, $\mathcal{B} = Mod(\mathbb{S})$ for κ-geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ-presentable objects. Suppose we have

$$\begin{tikzcd}
 & \mathcal{A} \arrow[r, "F"] & \mathcal{B} & \text{preserving } \kappa\text{-filtered colimits} \\
\text{...and restricting to} & \mathcal{A}_\kappa \arrow[u, hook] \arrow[r, "F_\kappa"] & \mathcal{B}_\kappa \arrow[u, hook] &
\end{tikzcd}$$

Then the following hold:

(a) If F_κ is essentially surjective, then $Th_{\kappa\text{-geom}}(F(\mathcal{A})) = Th_{\kappa\text{-geom}}(\mathcal{B})$.

Example

Example 1: (Dilworth's congruence lattice problem)

$$\begin{array}{ccc} \mathbf{Lat} & \to & \mathbf{DistSemLat} \\ L & \mapsto & Con_c(L) \end{array}$$

preserves $\aleph_1$-filtered colimits and $\aleph_1$-presentable objects.

Theorem (Huhn 1985)**:** Every DistSemLat of cardinality $\leq \aleph_1$ is of the form $Con_c(L)$.

Corollary: $\nexists$ an $\aleph_1$-geometric sequent that holds for all $Con_c(L)$ but not for general DistSemLats.

The result

Theorem (A.): Let $\mathcal{A}$, $\mathcal{B}$ be κ-accessible categories, $\mathcal{A} = Mod(\mathbb{T})$, $\mathcal{B} = Mod(\mathbb{S})$ for κ-geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ-presentable objects. Suppose we have

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Then the following hold:

(a) If F_κ is essentially surjective, then $Th_{\kappa\text{-geom}(F(\mathcal{A}))} = Th_{\kappa\text{-geom}(\mathcal{B})}$.

More generally: For different such functors $F\colon \mathcal{A} \to \mathcal{B}$, $F'\colon \mathcal{A}' \to \mathcal{B}$ one has $Th_{\kappa\text{-geom}}(F(\mathcal{A})) = Th_{\kappa\text{-geom}}(F'(\mathcal{A}')) \supseteq \mathbb{S}$ if and only if $F_\kappa(\mathcal{A})$ and $F'_\kappa(\mathcal{A}')$ have equivalent idempotent completions.

The result

Theorem (A.)(continued)**:** Let $\mathcal{A}$, $\mathcal{B}$ be κ-accessible categories, $\mathcal{A} = \mathit{Mod}(\mathbb{T})$, $\mathcal{B} = \mathit{Mod}(\mathbb{S})$ for κ-geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ-presentable objects. Suppose we have $F\colon \mathcal{A} \to \mathcal{B}$ preserving κ-filtered colimits and κ-presentable objects.

Then the following hold:

(b) If $F_\kappa\colon \mathcal{A}_\kappa \to \mathcal{B}_\kappa$ is *fully faithful*, then $F(\mathcal{A}) = \mathit{Mod}(\mathbb{S}')$ for some axiomatic extension $\mathbb{S}' \supseteq \mathbb{S}$ (i.e. the essential image $F(\mathcal{A})$ can be characterized by additional κ-geometric sequents in the language of $\mathbb{S}$).

(c) If one has that every $B \in \mathcal{B}_\kappa$ admits a morphism to $F(A)$, for some $A \in \mathcal{A}_\kappa$, then $\mathit{Th}_{\neg\text{-}\kappa\text{-geom}}(F(\mathcal{A})) = \mathit{Th}_{\neg\text{-}\kappa\text{-geom}}(\mathcal{B})$, i.e. the objects in the essential image of F and general objects of $\mathcal{B}$ satisfy exactly the same negations of κ-geometric formulas.

Examples

Example 2: (toy problem)

$$\begin{array}{rcl} \textbf{Rings w/ unit} & \to & \textbf{Ab} \\ (R,+,\cdot) & \mapsto & (R,+) \end{array}$$

Examples

Example 3: (representation problem for special groups; jt. with Hugo Mariano)

$$\begin{array}{rcl} \textbf{Fields} & \to & \textbf{SpecialGroups} \\ F & \mapsto & (F^*/(F^*)^2, \cdot, \Gamma_+) \end{array}$$

Questions (Marshall 1970s, Dickmann/Miraglia 1990s) **:**
– Is every special group isomorphic to one coming from a field?
– Is every special group elementarily equivalent to one coming from a field?

Theorem (Marshall 1970s)**:** Every *finite* special group is isomorphic to the special group of a field.

Theorem (Dickmann/Miraglia)**:** Every special group can be *embedded into* the special group of a von Neumann regular ring (= limit of fields).

Examples

Example 3: (representation problem for special groups; jt. with Hugo Mariano)

$$\begin{array}{rcl} \textbf{Fields} & \to & \textbf{SpecialGroups} \\ F & \mapsto & (F^*/(F^*)^2, \cdot, \Gamma_+) \end{array}$$

This functor preserves $\aleph_0$- and $\aleph_1$-directed colimits. It does not preserve $\aleph_0$-presentable objects, but $\aleph_1$-presentable objects.

Corollary of (c): General special groups satisfy the same negations of κ-geometric formulas as special groups of regular rings.

Consequence of (a): If special groups of fields can be distinguished by a $\aleph_1$-geometric formula, then there is a non-representable special group of cardinality $\aleph_1$.

About the proof

$$\begin{tikzcd}
 & \mathit{Toposes} & \\
\mathit{TopSpaces} \arrow[ur] & & \mathit{GeomTheories} \arrow[ul, two heads]
\end{tikzcd}$$

About the proof

$$\begin{tikzcd}
 & \mathit{Toposes} & \\
 & \rotatebox{90}{$\in$} & \\
\mathit{TopSpaces} \arrow[uur] & \mathit{Set}[\mathbb{T}] & \mathit{GeomTheories} \arrow[uul, two heads] \\
 & & \rotatebox{90}{$\in$} \\
 & & \mathbb{T} \arrow[uul, mapsto]
\end{tikzcd}$$

About the proof

$$\begin{tikzcd}
 & \mathit{Toposes} & \\
 & \rotatebox[origin=c]{90}{$\in$} & \\
\mathit{TopSpaces} \arrow[uur] & \mathit{Sh}(X) & \mathit{GeomTheories} \arrow[uul, two heads] \\
\rotatebox[origin=c]{90}{$\in$} & & \\
X \arrow[uur, mapsto] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \mathit{Toposes} & \\
 & \rotatebox[origin=c]{90}{$\in$} & \\
\mathit{TopSpaces} \arrow[uur] & Sh(X) \arrow[dd, shift right=2, "f_*"'] \arrow[dd, phantom, "\vdash"] & \mathit{GeomTheories} \arrow[uul, two heads] \\
\rotatebox[origin=c]{90}{$\in$} & & \\
X \arrow[uur, mapsto] \arrow[d, "f"'] & Sh(Y) \arrow[uu, shift right=2, "f^*"'] & \\
Y \arrow[ur, mapsto] & &
\end{tikzcd}$$

For good enough spaces:

– f surjective $\Leftrightarrow$ f^* faithful

– f embedding $\Leftrightarrow$ f_* fully faithful

– $f(X)$ dense in Y $\Leftrightarrow$ $f_*(0) \cong 0$

– f closed inclusion $\Leftrightarrow$ $f^*(G) \cong G \times U$ for a subterminal object U

$$\begin{tikzcd}
 & \mathit{Toposes} & \\
\mathit{TopSpaces} \arrow[ur] & Sh(X) \arrow[u, phantom, sloped, "\in"] \arrow[d, shift right=2, "f_*"'] \arrow[d, phantom, "\vdash"] & \mathit{GeomTheories} \arrow[ul, two heads] \\
X \arrow[u, phantom, sloped, "\in"] \arrow[ur, mapsto] \arrow[d, "f"'] & Sh(Y) \arrow[u, shift right=2, "f^*"'] & \\
Y \arrow[ur, mapsto] & &
\end{tikzcd}$$

For good enough spaces:

- f surjective $\Leftrightarrow f^*$ faithful $\Leftrightarrow$: (f_*, f^*) is a surjective geometric morphism
- f embedding $\Leftrightarrow f_*$ fully faithful $\Leftrightarrow$: (f_*, f^*) is an inclusion
- $f(X)$ dense in $Y \Leftrightarrow f_*(0) \cong 0$ $\Leftrightarrow$: (f_*, f^*) is dominant
- f closed inclusion $\Leftrightarrow f^*(G) \cong G \times U$ $\Leftrightarrow$: (f_*, f^*) is a closed inclusion

Factorization

In TopSpaces:

$$\begin{tikzcd}
X \arrow[rrr, "f"] \arrow[dr, two heads] & & & Y \\
& f(X) \arrow[r, hook] & \overline{f(X)} \arrow[ur, hook] &
\end{tikzcd}$$

In Toposes:

$$\begin{tikzcd}
\mathcal{E} \arrow[rrr, "(f_*{,}f^*)"] \arrow[dr, two heads] & & & \mathcal{F} \\
& \mathcal{F}' \arrow[r, hook] & \mathcal{F}'' \arrow[ur, hook] &
\end{tikzcd}$$

Factorization

In Toposes:

$$\begin{tikzcd}
\mathcal{E} \arrow[rrr, "{(f_*,f^*)}"] \arrow[dr, two heads] & & & \mathcal{F} \\
& \mathcal{F}' \arrow[r, hook] & \mathcal{F}'' \arrow[ur, hook] &
\end{tikzcd}$$

For classifying toposes:

$$\begin{tikzcd}
Set[\mathbb{T}] \arrow[rrr, "{(f_*,f^*)}"] \arrow[dr, two heads] & & & Set[\mathbb{S}] \\
& Set[\mathbb{S}'] \arrow[r, hook] & Set[\mathbb{S}''] \arrow[ur, hook] &
\end{tikzcd}$$

$\mathbb{S}' \supseteq \mathbb{S}'' \supseteq \mathbb{S}$ axiomatic extensions over the same signature.

Factorization

For classifying toposes:

$$\begin{tikzcd}
f^*(M_\mathbb{S}) & & & M_\mathbb{S} \arrow[lll, mapsto] \\
\mathit{Set}[\mathbb{T}] \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[rrr, "{(f_*,f^*)}"] \arrow[dr] & & & \mathit{Set}[\mathbb{S}] \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \\
& \mathit{Set}[\mathbb{S}'] \arrow[r, hook] & \mathit{Set}[\mathbb{S}''] \arrow[ur, hook] &
\end{tikzcd}$$

$\mathbb{S}' \supseteq \mathbb{S}'' \supseteq \mathbb{S}$ axiomatic extensions over the same signature, namely:

$\mathbb{S}' = \{$geometric sequents satisfied by $f^*(M_\mathbb{S})\} = Th_{geom}(f^*(M_\mathbb{S}))$

$\mathbb{S}'' = \{$negations of geometric formulas satisfied by $f^*(M_\mathbb{S})\} \cup \mathbb{S}$
$\quad = Th_{\neg\text{-}geom}(f^*(M_\mathbb{S})) \cup \mathbb{S}$

see [Caramello, Toposes, Sites and Theories]

κ-geometric morphisms

Defintion: A κ*-geometric morphism* is a geometric morphism (f_*, f^*) such that f^* preserves κ-small limits.

A κ-geometric morphism preserves all κ-geometric sequents.

Theory of κ-toposes and κ-geometric morphisms elaborated by **Christian Espíndola**: Deligne complenetess, Omitting types, ...

Proposition(A.)**:** For a κ-geometric morphism (f_*, f^*) the above factorization yields κ-geometric morphisms.

How to compute the factorization?

A functor between small categories $f\colon C \to D$ yields a κ-geometric morphism

$$\begin{tikzcd} Set^C \arrow[r, shift left, "Ran_f"] & Set^D \arrow[l, shift left, "- \circ f"] \end{tikzcd}$$

Proposition (joint w/ Eduardo Ochs)**:** For this geometric morphism the factorization is induced by a factorization of f:

$$\begin{tikzcd} C \arrow[rrr, "f"] \arrow[dr] & & & D \\ & D' \arrow[r, hook] & D'' \arrow[ur, hook] & \end{tikzcd}$$

where
– D' is the full subcategory of D whose objects are in the image of f
– D'' is the full subcategory of D whose objects admit a morphism into the image of f

Thus we get:

$$\begin{tikzcd}
Set^{C} \arrow[rrr, "f"] \arrow[dr, two heads] & & & Set^{D} \\
& Set^{D'} \arrow[r, hook] & Set^{D''} \arrow[ur, hook] &
\end{tikzcd}$$

where

– D' is the full subcategory of D whose objects are in the image of f

– D'' is the full subcategory of D whose objects admit a morphism into the image of f

About the proof

Relation to the situation of the theorem:

Fact: A κ-accessible category $\mathcal{A}$ is the category of *Set*-valued models of the κ-geometric theory $\mathbb{T}$ classified by the topos $Set^{\mathcal{A}_\kappa}$:

$$\mathcal{A} \simeq Mod(\mathbb{T}) \simeq \kappa\text{-}geom(Set, Set^{\mathcal{A}_\kappa})$$

- $Set[\mathbb{T}]_\kappa \simeq Set^{\mathcal{A}_\kappa}$ is a presheaf category

The hypotheses ensure that the functor $F\colon \mathcal{A} = Mod(\mathbb{T}) \to Mod(\mathbb{S}) = \mathcal{B}$ is induced by composing with a κ-geometric morphism

$$Set[\mathbb{T}]_\kappa := Set^{\mathcal{A}_\kappa} \to Set^{\mathcal{B}_\kappa} =: Set[\mathbb{S}]_\kappa$$

which is essential, induced by $F_\kappa\colon \mathcal{A}_\kappa \to \mathcal{B}_\kappa$

About the proof

$$\mathcal{A} \simeq \mathit{Mod}(\mathbb{T}) \simeq \kappa\text{-}\mathit{geom}(\mathit{Set}, \mathit{Set}^{\mathcal{A}_\kappa^{op}})$$

Factorize this morphism as

$$\begin{tikzcd}
\mathit{Set} \arrow[d] & & & \\
\mathit{Set}[\mathbb{T}]_\kappa \arrow[rrr, "{(f_*, f^*)}"] \arrow[dr, two heads] & & & \mathit{Set}[\mathbb{S}]_\kappa \\
& \mathit{Set}[\mathbb{S}']_\kappa \arrow[r, hook] & \mathit{Set}[\mathbb{S}'']_\kappa \arrow[ur, hook] &
\end{tikzcd}$$

About the proof

$$\begin{tikzcd}
Set[\mathbb{T}] \arrow[rrr, "(f_*{,}f^*)"] \arrow[dr] & & & Set[\mathbb{S}] \\
& Set[\mathbb{S}'] \arrow[r] & Set[\mathbb{S}''] \arrow[ur] &
\end{tikzcd}$$

where $\mathbb{S}' \supseteq \mathbb{S}'' \supseteq \mathbb{S}$ are axiomatic extensions over the same signature, namely: $\mathbb{S}' := Th_{\kappa\text{-}geom}(F(\mathcal{A}))$, $\mathbb{S}'' := Th_{\neg\text{-}\kappa\text{-}geom}(F(\mathcal{A})) \cup \mathbb{S}$

The conditions on F_κ ensure

– in case **(a)**: that the 2nd and 3rd morphisms are equivalences
– in case **(b)**: that the 1st morphism is an equivalence
– in case **(c)**: that the 3rd morphism is an equivalence

About the proof

This factorization becomes computable, and the conditions on F_κ exploitable, because

1. the toposes are *presheaf toposes*
2. the geometric morphisms are "essential"
   (i.e. induced by functors between the index categories)

Both 1. and 2. are made possible by passage from geometric to κ-geometric morphisms!

Example

Example 2: $\kappa = \aleph_0$, $\mathcal{B} =$ **Groups**, $\mathcal{A} = Mod(Th_{geom}(F(n)))$ where $F(n)$ is the free group on n generators.

Theorem (Kharlampovich-Myasnikov/Sela 2006)**:** The first order theories of all finitely generated free groups coincide.

Not true in $L_{\infty,\omega}$*:* $\exists x_1, \ldots, x_n \, \forall y \; \bigvee_{w \in words(x_1,\ldots,x_n)} y = w$.

Proposition: For $m < n$ we have $Th_{geom}(F(m)) \neq Th_{geom}(F(n))$.
Proof: Idempotent closure of $F(n)$ is $\{F(1), \ldots, F(n))\}$ - these are different for different n. □

Actually for $m < n$ we have $Th_{geom}(F(n)) \subsetneq Th_{geom}(F(m))$.

Afterthoughts

Continuations:

- applications
- exploit other factorizations
- ∞-categorical version
- relation to Wehrung's work?

Thank you!