

Ranges of functors and elementary classes via topos theory

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Toposes online, 29.6.2021

\mathbb{S}, \mathbb{T} first order theories, $F: \mathcal{A} := \text{Mod}(\mathbb{T}) \rightarrow \text{Mod}(\mathbb{S}) =: \mathcal{B}$ a functor.

Questions:

- ❶ Is every $B \in \mathcal{B}$ isomorphic to $F(A)$ for some $A \in \mathcal{A}$?
 - ❷ Is every $B \in \mathcal{B}$ elementarily equivalent to $F(A)$ for some $A \in \mathcal{A}$?
 - ❸ What is the elementary class generated by the essential image of F ?
- $$:= \text{Mod}(\bigcap_{A \in \mathcal{A}} \text{Th}(F(A)))$$

Rephrasing of (iii): Is there some first order statement true for every $F(A)$, but not for all B s ?

Example 1: Dilworth's congruence lattice problem

$$\begin{aligned} \mathbf{Lat} &\rightarrow \mathbf{AlgDistLat} \\ L &\mapsto \mathit{Con}(L) \end{aligned}$$

Question (Dilworth 1940s) : Is every $\mathbf{AlgDistLat}$ of the form $\mathit{Con}(L)$?

Example 1: Dilworth's congruence lattice problem

$$\begin{array}{lcl} \mathbf{Lat} & \rightarrow & \mathbf{DistSemLat} \\ L & \mapsto & \mathit{Con}_c(L) \end{array}$$

Question: Is every $\mathit{DistSemLat}$ of the form $\mathit{Con}_c(L)$?

Example 1: Dilworth's congruence lattice problem

$$\begin{array}{lcl} \mathbf{Lat} & \rightarrow & \mathbf{DistSemLat} \\ L & \mapsto & Con_c(L) \end{array}$$

Question: Is every DistSemLat of the form $Con_c(L)$?

Theorem (Huhn 1985): Every DistSemLat of cardinality $\leq \aleph_1$ is of the form $Con_c(L)$.

Theorem (Wehrung/Růžička 2007/8): There are DistSemLats of cardinality $\geq \aleph_2$ not of the form $Con_c(L)$.

Corollary: \nexists first order sentence holding for all $Con_c(L)$, but not for general DistSemLats.

My motivating examples:

- Representation problem for special groups (quadratic form theory)
- Which graded $\mathbb{Z}/2$ -algebras arise as Milnor K-theory of a field? (possible applications to inverse Galois problem)

What are *your* examples?

Definition: Let κ be a regular cardinal, Σ a first order signature.

- (i) A κ -geometric formula is a formula built from atomic formulas, \top, \perp , using $\bigvee_{j \in J}$ (J a set), $\bigwedge_{i \in I}$ ($|I| < \kappa$) and $\exists\{x_i\}_{i \in I}$ ($|I| < \kappa$).
- (ii) A κ -geometric theory is a theory which can be axiomatized by formulas of the form $\forall\{x_i\} \phi \rightarrow \psi$, where ϕ, ψ are κ -geometric formulas.
(κ -geometric sequents)
- (iii) For a class of Σ -structures \mathcal{C} denote by $Th_{\kappa\text{-geom}}(\mathcal{C})$ the κ -geometric theory of \mathcal{C} , i.e. the set of all κ -geometric sequents that are valid in every member of \mathcal{C} .
- (iv) Denote by $Th_{\neg\kappa\text{-geom}}(\mathcal{C})$ the set of negations of κ -geometric formulas (i.e. sequents of the form $\forall\bar{x} \phi \rightarrow \perp$), that are valid in every member of \mathcal{C} .

Reminder: Let κ be a regular cardinal.

(i) An object A is κ -presentable if for all κ -directed diagrams D

$$\text{Hom}(A, \text{colim } D) \cong \text{colim}_{d \in \text{Ob } D} \text{Hom}(A, d)$$

(ii) A category is κ -accessible if all κ -directed colimits exist and there is a set of κ -presentable objects, s.t. every object is a κ -directed colimit of these.

Examples:

- Any variety is \aleph_0 -accessible. (“finitely accessible”)
- **Fields** is \aleph_0 -accessible
- **Theorem:** The category $\text{Mod}(\mathbb{T})$ for a geometric theory \mathbb{T} is accessible.
- **Theorem:** The κ -accessible categories are exactly the ones of the form $\text{Mod}(\mathbb{T})$ for a κ -geometric theory \mathbb{T} .

The result

Theorem (A.): Let \mathcal{A}, \mathcal{B} be κ -accessible categories, $\mathcal{A} = \text{Mod}(\mathbb{T})$, $\mathcal{B} = \text{Mod}(\mathbb{S})$ for κ -geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ -presentable objects. Suppose we have

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\ \uparrow & & \uparrow \\ \mathcal{A}_\kappa & \xrightarrow{F_\kappa} & \mathcal{B}_\kappa \end{array} \quad \text{preserving } \kappa\text{-filtered colimits}$$

...and restricting to

Then the following hold:

(a) If F_κ is essentially surjective, then $\text{Th}_{\kappa\text{-geom}}(F(\mathcal{A})) = \text{Th}_{\kappa\text{-geom}}(\mathcal{B})$.

Example 1: (Dilworth's congruence lattice problem)

$$\begin{aligned} \mathbf{Lat} &\rightarrow \mathbf{DistSemLat} \\ L &\mapsto \mathit{Con}_c(L) \end{aligned}$$

preserves \aleph_1 -filtered colimits and \aleph_1 -presentable objects.

Theorem (Huhn 1985): Every $\mathbf{DistSemLat}$ of cardinality $\leq \aleph_1$ is of the form $\mathit{Con}_c(L)$.

Corollary: \nexists an \aleph_1 -geometric sequent that holds for all $\mathit{Con}_c(L)$ but not for general $\mathbf{DistSemLats}$.

The result

Theorem (A.): Let \mathcal{A}, \mathcal{B} be κ -accessible categories, $\mathcal{A} = \text{Mod}(\mathbb{T})$, $\mathcal{B} = \text{Mod}(\mathbb{S})$ for κ -geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ -presentable objects. Suppose we have

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Then the following hold:

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More generally: For different such functors $F: \mathcal{A} \rightarrow \mathcal{B}$, $F': \mathcal{A}' \rightarrow \mathcal{B}$ one has $\text{Th}_{\kappa\text{-geom}}(F(\mathcal{A})) = \text{Th}_{\kappa\text{-geom}}(F'(\mathcal{A}')) \supseteq \mathbb{S}$ if and only if $F_\kappa(\mathcal{A})$ and $F'_\kappa(\mathcal{A}')$ have equivalent idempotent completions.

Theorem (A.)(continued): Let \mathcal{A}, \mathcal{B} be κ -accessible categories, $\mathcal{A} = \text{Mod}(\mathbb{T}), \mathcal{B} = \text{Mod}(\mathbb{S})$ for κ -geometric theories. Denote by $\mathcal{A}_\kappa, \mathcal{B}_\kappa$ the subcategories of κ -presentable objects. Suppose we have $F: \mathcal{A} \rightarrow \mathcal{B}$ preserving κ -filtered colimits and κ -presentable objects.

Then the following hold:

- (b)** If $F_\kappa: \mathcal{A}_\kappa \rightarrow \mathcal{B}_\kappa$ is *fully faithful*, then $F(\mathcal{A}) = \text{Mod}(\mathbb{S}')$ for some axiomatic extension $\mathbb{S}' \supseteq \mathbb{S}$ (i.e. the essential image $F(\mathcal{A})$ can be characterized by additional κ -geometric sequents in the language of \mathbb{S}).
- (c)** If one has that every $B \in \mathcal{B}_\kappa$ admits a morphism to $F(A)$, for some $A \in \mathcal{A}_\kappa$, then $\text{Th}_{\neg\kappa\text{-geom}}(F(\mathcal{A})) = \text{Th}_{\neg\kappa\text{-geom}}(\mathcal{B})$, i.e. the objects in the essential image of F and general objects of \mathcal{B} satisfy exactly the same negations of κ -geometric formulas.

Example 2: (toy problem)

$$\begin{array}{lcl} \mathbf{Rings\ w/\ unit} & \rightarrow & \mathbf{Ab} \\ (R, +, \cdot) & \mapsto & (R, +) \end{array}$$

Example 3: (representation problem for special groups; jt. with Hugo Mariano)

$$\begin{aligned} \mathbf{Fields} &\rightarrow \mathbf{SpecialGroups} \\ F &\mapsto (F^*/(F^*)^2, \cdot, \Gamma_+) \end{aligned}$$

Questions (Marshall 1970s, Dickmann/Miraglia 1990s) :

- Is every special group isomorphic to one coming from a field?
- Is every special group elementarily equivalent to one coming from a field?

Theorem (Marshall 1970s): Every *finite* special group is isomorphic to the special group of a field.

Theorem (Dickmann/Miraglia): Every special group can be *embedded into* the special group of a von Neumann regular ring (= limit of fields).

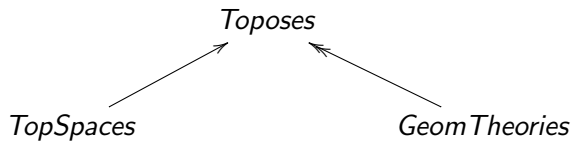
Example 3: (representation problem for special groups; jt. with Hugo Mariano)

$$\begin{aligned} \mathbf{Fields} &\rightarrow \mathbf{SpecialGroups} \\ F &\mapsto (F^*/(F^*)^2, \cdot, \Gamma_+) \end{aligned}$$

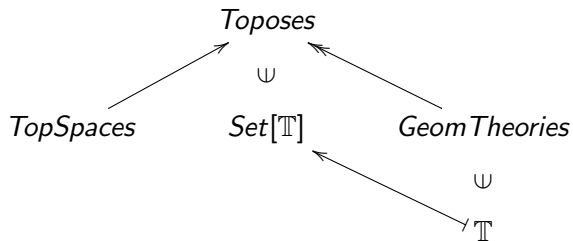
This functor preserves \aleph_0 - and \aleph_1 -directed colimits. It does not preserve \aleph_0 -presentable objects, but \aleph_1 -presentable objects.

Corollary of (c): General special groups satisfy the same negations of κ -geometric formulas as special groups of regular rings.

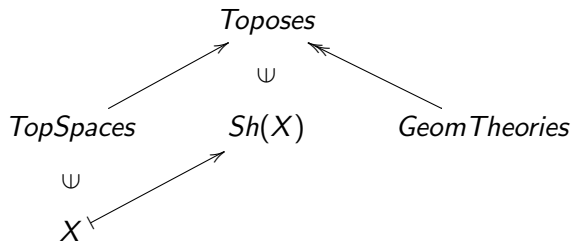
Consequence of (a): If special groups of fields can be distinguished by a \aleph_1 -geometric formula, then there is a non-representable special group of cardinality \aleph_1 .

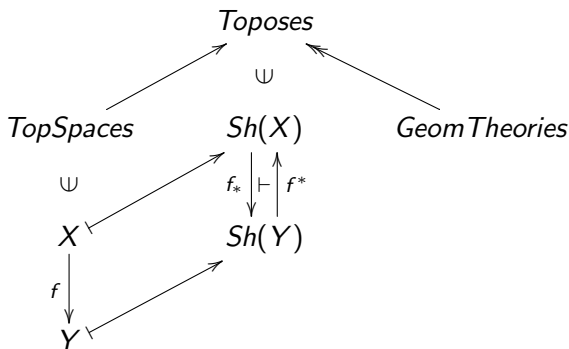


About the proof



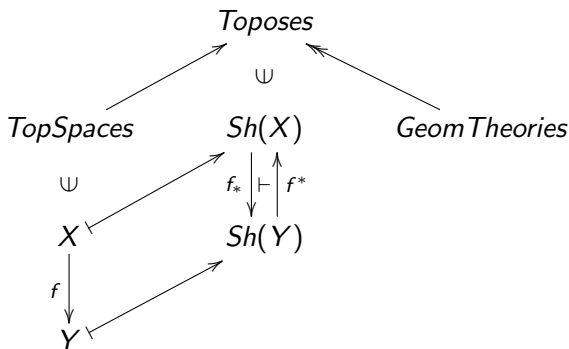
About the proof





For good enough spaces:

- f surjective $\Leftrightarrow f^*$ faithful
- f embedding $\Leftrightarrow f_*$ fully faithful
- $f(X)$ dense in $Y \Leftrightarrow f_*(0) \cong 0$
- f closed inclusion $\Leftrightarrow f^*(G) \cong G \times U$ for a subterminal object U



For good enough spaces:

- f surjective $\Leftrightarrow f^*$ faithful $\Leftrightarrow (f_*, f^*)$ is a surjective geometric morphism
- f embedding $\Leftrightarrow f_*$ fully faithful $\Leftrightarrow (f_*, f^*)$ is an inclusion
- $f(X)$ dense in $Y \Leftrightarrow f_*(0) \cong 0 \Leftrightarrow (f_*, f^*)$ is dominant
- f closed inclusion $\Leftrightarrow f^*(G) \cong G \times U \Leftrightarrow (f_*, f^*)$ is a closed inclusion

In TopSpaces:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \nearrow \\ & f(X) \hookrightarrow \overline{f(X)} & \end{array}$$

In Toposes:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{(f_*, f^*)} & \mathcal{F} \\ & \searrow & \nearrow \\ & \mathcal{F}' \hookrightarrow \mathcal{F}'' & \end{array}$$

In Toposes:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{(f_*, f^*)} & \mathcal{F} \\ & \searrow & \nearrow \\ & \mathcal{F}' \hookrightarrow \mathcal{F}'' & \end{array}$$

For classifying toposes:

$$\begin{array}{ccc} \text{Set}[\mathbb{T}] & \xrightarrow{(f_*, f^*)} & \text{Set}[\mathbb{S}] \\ & \searrow & \nearrow \\ & \text{Set}[\mathbb{S}'] \hookrightarrow \text{Set}[\mathbb{S}''] & \end{array}$$

$\mathbb{S}' \supseteq \mathbb{S}'' \supseteq \mathbb{S}$ axiomatic extensions over the same signature.

For classifying toposes:

$$\begin{array}{ccc}
 f^*(M_{\mathbb{S}}) & \longleftarrow & M_{\mathbb{S}} \\
 \cap & & \cap \\
 \text{Set}[\mathbb{T}] & \xrightarrow{(f_*, f^*)} & \text{Set}[\mathbb{S}] \\
 \searrow & & \nearrow \\
 & \text{Set}[\mathbb{S}'] \hookrightarrow \text{Set}[\mathbb{S}''] &
 \end{array}$$

$\mathbb{S}' \supseteq \mathbb{S}'' \supseteq \mathbb{S}$ axiomatic extensions over the same signature, namely:

$$\mathbb{S}' = \{\text{geometric sequents satisfied by } f^*(M_{\mathbb{S}})\} = Th_{\text{geom}}(f^*(M_{\mathbb{S}}))$$

$$\begin{aligned}
 \mathbb{S}'' &= \{\text{negations of geometric formulas satisfied by } f^*(M_{\mathbb{S}})\} \cup \mathbb{S} \\
 &= Th_{\neg\text{-geom}}(f^*(M_{\mathbb{S}})) \cup \mathbb{S}
 \end{aligned}$$

see [Caramello, Toposes, Sites and Theories]

κ -geometric morphisms

Defintion: A κ -geometric morphism is a geometric morphism (f_*, f^*) such that f^* preserves κ -small limits.

A κ -geometric morphism preserves all κ -geometric sequents.

Theory of κ -toposes and κ -geometric morphisms elaborated by **Christian Espíndola**: Deligne completeness, Omitting types, ...

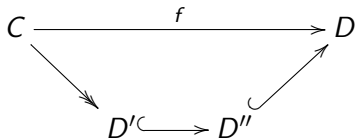
Proposition(A.): For a κ -geometric morphism (f_*, f^*) the above factorization yields κ -geometric morphisms.

How to compute the factorization?

A functor between small categories $f: C \rightarrow D$ yields a κ -geometric morphism

$$\text{Set}^C \begin{array}{c} \xrightarrow{\text{Ran}_f} \\ \xleftarrow{- \circ f} \end{array} \text{Set}^D$$

Proposition (joint w/ Eduardo Ochs): For this geometric morphism the factorization is induced by a factorization of f :



where

- D' is the full subcategory of D whose objects are in the image of f
- D'' is the full subcategory of D whose objects admit a morphism into the image of f

Thus we get:

$$\begin{array}{ccc} \text{Set}^C & \xrightarrow{f} & \text{Set}^D \\ & \searrow & \nearrow \\ & \text{Set}^{D'} \hookrightarrow \text{Set}^{D''} & \end{array}$$

where

- D' is the full subcategory of D whose objects are in the image of f
- D'' is the full subcategory of D whose objects admit a morphism into the image of f

Relation to the situation of the theorem:

Fact: A κ -accessible category \mathcal{A} is the category of Set -valued models of the κ -geometric theory \mathbb{T} classified by the topos $\text{Set}^{\mathcal{A}_\kappa}$:

$$\mathcal{A} \simeq \text{Mod}(\mathbb{T}) \simeq \kappa\text{-geom}(\text{Set}, \text{Set}^{\mathcal{A}_\kappa})$$

– $\text{Set}[\mathbb{T}]_\kappa \simeq \text{Set}^{\mathcal{A}_\kappa}$ is a presheaf category

The hypotheses ensure that the functor $F: \mathcal{A} = \text{Mod}(\mathbb{T}) \rightarrow \text{Mod}(\mathbb{S}) = \mathcal{B}$ is induced by composing with a κ -geometric morphism

$$\text{Set}[\mathbb{T}]_\kappa := \text{Set}^{\mathcal{A}_\kappa} \rightarrow \text{Set}^{\mathcal{B}_\kappa} =: \text{Set}[\mathbb{S}]_\kappa$$

which is essential, induced by $F_\kappa: \mathcal{A}_\kappa \rightarrow \mathcal{B}_\kappa$

About the proof

$$\mathcal{A} \simeq \text{Mod}(\mathbb{T}) \simeq \kappa\text{-geom}(\text{Set}, \text{Set}^{\mathcal{A}_{\kappa}^{\text{op}}})$$

Factorize this morphism as

A commutative diagram illustrating the factorization of a morphism. At the top left is the category Set . A vertical arrow points down to $\text{Set}[\mathbb{T}]_{\kappa}$. From $\text{Set}[\mathbb{T}]_{\kappa}$, a horizontal arrow points right to $\text{Set}[\mathbb{S}]_{\kappa}$, labeled (f_*, f^*) . Below $\text{Set}[\mathbb{T}]_{\kappa}$ and $\text{Set}[\mathbb{S}]_{\kappa}$ are two more categories: $\text{Set}[\mathbb{S}']_{\kappa}$ and $\text{Set}[\mathbb{S}'']_{\kappa}$. An arrow points from $\text{Set}[\mathbb{T}]_{\kappa}$ down to $\text{Set}[\mathbb{S}']_{\kappa}$. An arrow points from $\text{Set}[\mathbb{S}']_{\kappa}$ to $\text{Set}[\mathbb{S}'']_{\kappa}$, with a small circle above it indicating an inclusion. Finally, an arrow points from $\text{Set}[\mathbb{S}'']_{\kappa}$ up to $\text{Set}[\mathbb{S}]_{\kappa}$, with a small circle above it indicating an inclusion.

About the proof

$$\begin{array}{ccc} \text{Set}[\mathbb{T}] & \xrightarrow{(f_*, f^*)} & \text{Set}[\mathbb{S}] \\ & \searrow & \nearrow \\ & \text{Set}[\mathbb{S}'] \longrightarrow \text{Set}[\mathbb{S}''] & \end{array}$$

where $\mathbb{S}' \supseteq \mathbb{S}'' \supseteq \mathbb{S}$ are axiomatic extensions over the same signature, namely: $\mathbb{S}' := \text{Th}_{\kappa\text{-geom}}(F(\mathcal{A}))$, $\mathbb{S}'' := \text{Th}_{\neg\kappa\text{-geom}}(F(\mathcal{A})) \cup \mathbb{S}$

The conditions on F_κ ensure

- in case **(a)**: that the 2nd and 3rd morphisms are equivalences
- in case **(b)**: that the 1st morphism is an equivalence
- in case **(c)**: that the 3rd morphism is an equivalence

This factorization becomes computable, and the conditions on F_{κ} exploitable, because

1. the toposes are *presheaf toposes*
2. the geometric morphisms are “essential”
(i.e. induced by functors between the index categories)

Both 1. and 2. are made possible by passage from geometric to κ -geometric morphisms!

Example

Example 2: $\kappa = \aleph_0$, $\mathcal{B} = \mathbf{Groups}$, $\mathcal{A} = Mod(Th_{geom}(F(n)))$ where $F(n)$ is the free group on n generators.

Theorem (Kharlampovich-Myasnikov/Sela 2006): The first order theories of all finitely generated free groups coincide.

Not true in $L_{\infty, \omega}$: $\exists x_1, \dots, x_n \forall y \bigvee_{w \in words(x_1, \dots, x_n)} y = w$.

Proposition: For $m < n$ we have $Th_{geom}(F(m)) \neq Th_{geom}(F(n))$.

Proof: Idempotent closure of $F(n)$ is $\{F(1), \dots, F(n)\}$ - these are different for different n . \square

Actually for $m < n$ we have $Th_{geom}(F(n)) \subsetneq Th_{geom}(F(m))$.

Continuations:

- applications
- exploit other factorizations
- ∞ -categorical version
- relation to Wehrung's work?

Thank you!